Springer Monographs in Mathematics

Ivan Penkov
Crystal Hoyt

Classical Lie Algebras at Infinity

Springer

Springer Monographs in Mathematics

Editors-in-Chief

Minhyong Kim, Coventry, School of Mathematics, Korea Institute for Advanced Study, Seoul, South Korea

Mathematical Institute, University of Warwick, Coventry, UK

Katrin Wendland, Research group for Mathematical Physics, Albert Ludwigs University of Freiburg, Freiburg, Germany

Series Editors

Sheldon Axler, Department of Mathematics, San Francisco State University, San Francisco, CA, USA

Mark Braverman, Department of Mathematics, Princeton University, Princeton, NY, USA

Maria Chudnovsky, Department of Mathematics, Princeton University, Princeton, NY, USA

Tadahisa Funaki, Department of Mathematics, University of Tokyo, Tokyo, Japan

Isabelle Gallagher, Département de Mathématiques et Applications, Ecole Normale Supérieure, Paris, France

Sinan Güntürk, Courant Institute of Mathematical Sciences, New York University, New York, NY, USA

Claude Le Bris, CERMICS, Ecole des Ponts ParisTech, Marne la Vallée, France

Pascal Massart, Département de Mathématiques, Université de Paris-Sud, Orsay, France

Alberto A. Pinto, Department of Mathematics, University of Porto, Porto, Portugal

Gabriella Pinzari, Department of Mathematics, University of Padova, Padova, Italy

Ken Ribet, Department of Mathematics, University of California, Berkeley, CA, USA

René Schilling, Institute for Mathematical Stochastics, Technical University Dresden, Dresden, Germany

Panagiotis Souganidis, Department of Mathematics, University of Chicago, Chicago, IL, USA

Endre Süli, Mathematical Institute, University of Oxford, Oxford, UK

Shmuel Weinberger, Department of Mathematics, University of Chicago, Chicago, IL, USA

Boris Zilber, Mathematical Institute, University of Oxford, Oxford, UK

This series publishes advanced monographs giving well-written presentations of the "state-of-the-art" in fields of mathematical research that have acquired the maturity needed for such a treatment. They are sufficiently self-contained to be accessible to more than just the intimate specialists of the subject, and sufficiently comprehensive to remain valuable references for many years. Besides the current state of knowledge in its field, an SMM volume should ideally describe its relevance to and interaction with neighbouring fields of mathematics, and give pointers to future directions of research.

More information about this series at https://link.springer.com/bookseries/3733

Ivan Penkov • Crystal Hoyt

Classical Lie Algebras at Infinity

Springer

Ivan Penkov
Jacobs University Bremen
Bremen, Germany

Crystal Hoyt
Department of Mathematics
University of Haifa
Haifa, Israel

ISSN 1439-7382 ISSN 2196-9922 (electronic)
Springer Monographs in Mathematics
ISBN 978-3-030-89659-1 ISBN 978-3-030-89660-7 (eBook)
https://doi.org/10.1007/978-3-030-89660-7

Mathematics Subject Classification: 17B10, 17B55, 17B65

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2022
This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This book originated from graduate topics courses given by the first author at Yale University and at the University of California, Berkeley. Since then, the exposition has grown to include some recent research results in the field. Our book is intended as a bridge between the traditional university graduate courses and research-level mathematics in representation theory. It is therefore appropriate for both an advanced graduate student entering the field and a research mathematician wishing to diversify and expand their knowledge.

The unifying theme of this book is the structure and representation theory of infinite-dimensional locally reductive Lie algebras and superalgebras. The first six chapters are foundational, while each of the last four chapters represents a research specialization in this large field, and for the most part can be studied independently.

A locally reductive Lie algebra is a direct limit of finite-dimensional reductive Lie algebras. This property is an essential difference from the well-studied Kac–Moody algebras of finite rank (see [K5]), which have finite-dimensional Cartan subalgebras but are not direct limits of finite-dimensional Lie algebras (unless they happen to be finite dimensional). An important class of locally reductive Lie algebras, namely root-reductive Lie algebras, can be considered as Kac–Moody algebras of infinite rank; however, they do not come with a choice of simple roots. Moreover, the representation theory of these Lie algebras exploits essentially the fact that they admit triangular decompositions that are not of Kac–Moody type.

An outline of the book is as follows. Chapter 1 is a fast-paced review of the structure theory of finite-dimensional Lie algebras, as needed for later chapters. In Chap. 2, we discuss finite-dimensional Lie superalgebras with more attention to detail for the unfamiliar reader. In Chap. 3, we introduce the central object of this book: root-reductive Lie algebras, which are defined as direct limits of finite-dimensional reductive Lie algebras with compatible root decompositions. In Chap. 4, we consider two generalizations, namely classically semisimple infinite rank Lie superalgebras and a class of direct limits of finite-dimensional reductive Lie algebras in which the root decompositions are no longer compatible.

Chapters 5 and 6 are devoted to the structure theory of the most basic root-reductive Lie algebras: $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$. More precisely, we describe Cartan subalgebras, Borel subalgebras, and parabolic subalgebras in maximal generality. A key notion for this is the notion of a generalized flag. This notion is specific to the Lie algebras we study and is not borrowed from the theory of classical finite-dimensional Lie algebras or Kac–Moody algebras.

In Chap. 7, we turn our attention to the representation theory of the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$. The category of tensor modules is a very natural analog of the category of finite-dimensional representations of a classical Lie algebra (with the exception of spinor modules for $\mathfrak{o}(\infty)$). In contrast with the latter category, which is semisimple, tensor modules over $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$ form a non-semisimple Koszul category. We also briefly discuss tensor modules for Lie superalgebras.

In Chap. 8, we present some general results on weight modules. Since these results are not so widely known, even for finite-dimensional Lie algebras, we also discuss this case. For $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$, we pay special attention to recent results about simple bounded weight modules.

Chapter 9 is a brief review of generalized Harish-Chandra modules. These are modules with a locally finite action of a fixed subalgebra which, in contrast with the classical case considered by Harish-Chandra, does not have to be symmetric. Our main tools here are the Fernando–Kac subalgebra of a representation and the functor of cohomological induction, or the Zuckerman functor. Again, these results are not so widely known for finite-dimensional Lie algebras, and so we present them in some detail. In the case of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$, the theory of generalized Harish-Chandra modules has only made its first steps.

Finally, in Chap. 10, we turn to geometry. We take up a jewel in classical representation theory, namely the Bott–Borel–Weil theorem, and discuss its known analogs for the Lie supergroups and locally reductive ind-groups. For convenience, we begin with presenting an explicit version of Demazure's proof for the Lie group $GL(V)$.

The necessary background for Chaps. 1–6 is just a standard introductory course on Lie algebras and their representations, while Chaps. 7 and 9 require some homological algebra, and Chap. 10 assumes basic knowledge of algebraic geometry and sheaf cohomology. Lie superalgebras appear only in Chaps. 2, 4, 7, and 10 and do not require prior knowledge beyond Lie algebras. Most chapters should fill several hour-long sessions if one decided to give a course based on the book. Exercises of various levels of difficulty are interlaced throughout the book to add depth to reading comprehension.

We see our exposition as an invitation to a wide open research area in the form of a guide, and our objective would be more than achieved if we have succeeded in confronting the reader with a thought-provoking complex mixture of mathematical ideas.

Acknowledgments

The first author thanks Yale University and the University of California at Berkeley where a first version of this book originated. Both authors thank the Weizmann Institute where this book was expanded and brought into its current form. The second author thanks Jacobs University, Bremen, for hosting while working on the book.

We express our gratitude to Alexander Baranov, Ivan Dimitrov, Karl-Hermann Neeb, and Gregg Zuckerman for useful discussions of the emerging invariants of locally simple locally finite Lie algebras, and to Herbert Gangl, Georgi Simidchiev, and Bogdan Vioreanu for help in the exploration of the open problems concerning supernatural numbers related to $\mathfrak{sl}(i, \mathrm{adj})$. In addition, we thank Kevin Coulembier, Elizabeth Dan-Cohen, Ivan Dimitrov, Dimitar Grantcharov, Siarhei Markouski, Todor Milev, Alexander Voronov, and Pablo Zadunaisky for reading drafts of the manuscript and making comments that helped improve the presentation. We are grateful to Ann Kostant for valuable guidance during the process of preparing this book for publication. Last but not least, we thank Mel de Veccio and Todor Milev for typing assistance.

Bremen, Germany — Ivan Penkov
Haifa, Israel — Crystal Hoyt

Contents

Part I Structure of Locally Reductive Lie Algebras

1 Finite-Dimensional Lie Algebras 3
1.1 Lie Algebras 3
1.2 Cartan Subalgebras and Root Systems 5
1.3 Borel Subalgebras and Sets of Simple Roots 7

2 Finite-Dimensional Lie Superalgebras 11
2.1 Lie Superalgebras 11
2.2 Classical Lie Superalgebras 14
2.2.1 Definitions 14
2.2.2 Invariant Bilinear Forms 17
2.3 Simple Lie Superalgebras 18
2.4 Semisimple Lie Superalgebras 20
2.5 Cartan Subsuperalgebras and Root Systems 21
2.6 Borel Subsuperalgebras and Sets of Simple Roots 25

3 Root-Reductive Lie Algebras 31
3.1 Definitions 31
3.2 Locally Simple Root-Reductive Lie Algebras 33
3.3 Description of Root-Reductive Lie Algebras 36
3.4 Generalized Root Decomposition 37
3.5 Characterization of Root-Reductive Lie Algebras 38

4 Two Generalizations 41
4.1 Root-Reductive Lie Superalgebras 41
4.1.1 Definitions 41
4.1.2 Classically Semisimple Lie Superalgebras 42
4.1.3 The Finite-Dimensional Case 44
4.1.4 Classical Locally Simple Lie Superalgebras 45
4.1.5 Description of Classically Semisimple Lie Superalgebras 46

4.2 Locally Simple Lie Algebras 47
4.2.1 Diagonal Locally Simple Lie Algebras 47
4.2.2 Supernatural Numbers 48
4.2.3 Non-diagonal Locally Simple Lie Algebras 51

5 Splitting Borel Subalgebras of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ and Generalized Flags 55
5.1 Splitting Borel Subalgebras 55
5.1.1 Splitting Borel Subalgebras of $\mathfrak{sl}(\infty)$ 56
5.1.2 Splitting Borel Subalgebras of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ 61
5.2 Generalized Flags 62
5.2.1 Finite-Dimensional Flags 63
5.2.2 Generalized Flags 64
5.2.3 Maximal Chains and Parabolic Subalgebras of $\mathfrak{sl}(\infty)$ 66
5.2.4 The Case of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ 67

6 General Cartan, Borel and Parabolic Subalgebras of $\mathfrak{gl}(\infty)$ and $\mathfrak{sl}(\infty)$ 69
6.1 Cartan Subalgebras of $\mathfrak{sl}(\infty)$ and $\mathfrak{gl}(\infty)$ 69
6.1.1 Toral Subalgebras 70
6.1.2 Dual Systems 72
6.1.3 Cartan Subalgebras 77
6.2 Borel and Parabolic Subalgebras of $\mathfrak{sl}(\infty)$ and $\mathfrak{gl}(\infty)$ 80
6.3 Closing Remarks 84

Part II Modules over Locally Reductive Lie Algebras

7 Tensor Modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ 89
7.1 Structure of Mixed Tensor Products 89
7.2 The Category $\mathbb{T}_{\mathfrak{g}}$ 99
7.3 Injective Objects in $\mathbb{T}_{\mathfrak{g}}$ 105
7.4 Vanishing Theorem for Ext in $\mathbb{T}_{\mathfrak{g}}$ 107
7.5 Koszulity of the Category $\mathbb{T}_{\mathfrak{g}}$ 110
7.6 Koszul Self-duality of the Ring $\mathcal{A}_{\mathfrak{sl}(\infty)}$ 114
7.7 Equivalence of the Categories $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$ 119
7.8 Categories of Tensor Modules for Lie Superalgebras 121
7.8.1 The Case of $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$ 122
7.8.2 The Case of $\mathfrak{g} = \mathfrak{osp}(\infty|\infty)$ 123
7.9 Further Developments 123

8 Weight Modules 125
8.1 Preliminaries 125
8.2 Weight Modules for Finite-Dimensional Lie Algebras 128
8.2.1 The $\mathfrak{h}$-Support of a $(\mathfrak{g}, \mathfrak{h})$-Weight Module 128
8.2.2 The Shadow Decomposition 130
8.2.3 $(\mathfrak{g}, \mathfrak{h})$-Modules in a Special Case 131
8.2.4 General $(\mathfrak{g}, \mathfrak{h})$-Modules 133

8.2.5 Fundamental Results for Reductive $\mathfrak{g}$ 136
8.2.6 Cuspidal Modules and Mathieu's Coherent Families 138
8.2.7 Weight Modules for Lie Superalgebras 140
8.3 Weight Modules for Root-Reductive Lie Algebras 141
8.3.1 Main Differences from the Finite-Dimensional Case 141
8.3.2 The Shadow Decomposition 143
8.3.3 Simple Bounded Weight Modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ 145
8.3.4 Simple Highest Weight Modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ with Finite-Dimensional Weight Spaces ... 150
8.4 Categories of Modules over Root-Reductive Lie Algebras 155
8.4.1 Categories of Integrable Modules 155
8.4.2 Categories Containing Non-Integrable Modules 157
8.5 Closing Remarks 157

9 Generalized Harish-Chandra Modules 159
9.1 Introduction 159
9.2 The Case where $\mathfrak{g}$ is Finite Dimensional 162
9.2.1 Construction of $(\mathfrak{g}, \mathfrak{k})$-Modules 162
9.2.2 Reconstruction Theorems 168
9.2.3 Fernando–Kac Subalgebra of a Fundamental Series Module 173
9.2.4 The Case $\mathfrak{k} \simeq \mathfrak{sl}(2)$ 175
9.3 The Case when $\mathfrak{g}$ is Infinite Dimensional 177
9.4 Closing Remarks 183

Part III Geometric Aspects

10 The Bott–Borel–Weil Theorem 187
10.1 The Classical Bott–Borel–Weil Theorem 187
10.1.1 Introductory Considerations 188
10.1.2 The Main Result 193
10.2 Toward a Bott–Borel–Weil Theorem for Lie Supergroups 199
10.2.1 Preliminaries on Lie Supergroups and Flag Supermanifolds 199
10.2.2 Bott–Borel–Weil Theorem for Typical Weights 205
10.3 Toward a Bott-Borel-Weil Theorem for Locally Reductive Ind-Groups 211
10.3.1 The Case of a Line Bundle 211
10.3.2 The Case of Growing Rank 219
10.4 Open Problems 225

References 227

Index of Notation 233

Index 237

Notation and Terminology

- The ground field is $\mathbb{C}$. All vector spaces (including Lie algebras and superalgebras) are assumed countable dimensional, except in Sect. 7.9.
- Instead of $\mathbb{N}$ and $\mathbb{N} \setminus \{0\}$ we use, respectively, the unconventional but self-explanatory notations $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$.
- The sign $\subset$ stands for a set theoretic inclusion that is not necessarily strict.
- By a strict partial order or strict linear order $\prec$ on a set S, we mean a respective order such that $s_1 \prec s_2$ and $s_2 \prec s_1$ is a contradiction.
- The symbol $\bigsqcup$ stands for disjoint union.
- The superscript * indicates (algebraic) dual space, and span $=$ span$_{\mathbb{C}}$, $\otimes = \otimes_{\mathbb{C}}$.
- The symbol $\boxtimes$ denotes outer tensor product.
- The sign $\dotplus$ stands for semidirect sum of Lie algebras and superalgebras: in $\mathfrak{a} \dotplus \mathfrak{b}$, $\mathfrak{a}$ is an ideal and $\mathfrak{b}$ is in general merely a subalgebra. The sign $\oplus$ can mean a direct sum as Lie (super)algebras or as vector spaces, and should be understood from the context.
- Subscripts and superscripts: whenever we consider a $\mathbb{Z}_2$-graded vector space, we use the subscripts $\overline{0}$ and $\overline{1}$ to denote its even and odd parts, respectively. If a sequence of vector spaces is indexed by subscripts from $\mathbb{Z}_{>0}$, then no $\mathbb{Z}_2$-grading is implied.
- Sign convention: If x is an element of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ and the symbol $\bar{x}$ appears in an equation involving x, then x is assumed to be $\mathbb{Z}_2$-homogeneous and $\bar{x} \in \mathbb{Z}_2$ is the parity of x, i.e., $x \in \mathfrak{g}_{\bar{x}}$.
- If $\mathcal{C}$ is a category, then we write $X \in \mathcal{C}$ whenever X is an object of $\mathcal{C}$.

Part I
Structure of Locally Reductive Lie Algebras

In this part of the book, we develop the structure theory of locally reductive Lie algebras, including Cartan subalgebras, root systems, Borel subalgebras and parabolic subalgebras. A locally reductive Lie algebra is a direct limit of finite-dimensional reductive Lie algebras. Naturally, the techniques used to study locally reductive Lie algebras rely on the structure theory of finite-dimensional reductive Lie algebras.

Chapter 1 begins by recalling fundamental results about finite-dimensional Lie algebras. In particular, we define the classical Lie algebras $\mathfrak{gl}(V)$, $\mathfrak{sl}(V)$, $\mathfrak{o}(V)$, $\mathfrak{sp}(V)$. More generally, we recall the definitions of radical, semisimple part, Levi component, Cartan subalgebra, root system, simple roots, Weyl group and Borel subalgebra. We also list the root systems of all simple finite-dimensional Lie algebras.

A Lie superalgebra is a generalization of a Lie algebra in the setup of $\mathbb{Z}_2$-graded vector spaces. In Chap. 2, we recall the classification of finite-dimensional simple Lie superalgebras, with emphasis on the classical Lie superalgebras whose structure theory resembles that of finite-dimensional simple Lie algebras. We then discuss the structure of finite-dimensional semisimple Lie superalgebras (i.e., those with trivial radical) which is more complicated than that of semisimple Lie algebras. We define Cartan subsuperalgebra, root system, Weyl group and Borel subsuperalgebra. One new phenomenon is the existence of odd reflections which we introduce towards the end of the chapter.

The main topic of this book, namely locally reductive Lie algebras, is reached in Chap. 3. We start more generally with locally finite Lie algebras, which are arbitrary direct limits of finite-dimensional Lie algebras. Then we move to root-reductive Lie algebras and present two structure results for this class of infinite-dimensional Lie algebras.

Chapter 4 discusses two generalizations of locally simple root-reductive Lie algebras. The first one concerns a superalgebra analog of these Lie algebras: classically semisimple locally finite Lie superalgebras. The second one concerns general locally simple Lie algebras which are no longer root-reductive. Here supernatural numbers (or Steinitz numbers) come into play.

Chapters 5 and 6 are devoted to the structure of Cartan and Borel subalgebras of the Lie algebras $\mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. We start with discussing the most important class of splitting Borel subalgebras. We study the combinatorics of their roots and their simple roots. We then relate splitting Borel subalgebras to generalized flags in the natural representation.

A detailed discussion of general Cartan subalgebras is postponed to Chap. 6. Here we go beyond splitting Cartan subalgebras, in other words beyond direct limits of finite-dimensional Cartan subalgebras. Finally, in the second part of Chap. 6, we study general Borel and parabolic subalgebras of $\mathfrak{gl}(\infty)$ and $\mathfrak{sl}(\infty)$ that do not contain a splitting Cartan subalgebra.

Chapter 1
Finite-Dimensional Lie Algebras

In this chapter we recall some fundamental facts about finite-dimensional complex Lie algebras. All the pertinent proofs can be found in [B], and the readers are strongly encouraged to familiarize themselves with the relevant parts of this classical treatise, which is an ideal basic reference for the subject.

1.1 Lie Algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra (over $\mathbb{C}$). There is a canonical exact sequence of Lie algebras

$$0 \to \mathfrak{r}_{\mathfrak{g}} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}_{ss} \to 0, \tag{1.1}$$

where $\mathfrak{r}_{\mathfrak{g}}$ is the (unique) maximal solvable ideal in $\mathfrak{g}$ called the *solvable radical* (or simply the *radical*) of $\mathfrak{g}$. The quotient $\mathfrak{g}_{ss} = \mathfrak{g}/\mathfrak{r}_{\mathfrak{g}}$ is called the *semisimple part* of $\mathfrak{g}$. The Levi–Malcev Theorem [B, Ch.I, §6, no.8] claims that the exact sequence (1.1) splits. In general, there is no canonical splitting, and fixing a subalgebra $\mathfrak{g}_L$ of $\mathfrak{g}$ isomorphic to $\mathfrak{g}_{ss}$ is equivalent to fixing a splitting. Any such subalgebra $\mathfrak{g}_L$ of $\mathfrak{g}$ automatically satisfies $\mathfrak{g}_L \cap \mathfrak{r}_{\mathfrak{g}} = 0$ (prove!), and is called a *Levi component* of $\mathfrak{g}$.

The choice of a Levi component of $\mathfrak{g}$ determines an isomorphism of Lie algebras $\mathfrak{g} \simeq \mathfrak{r}_{\mathfrak{g}} \dotplus \mathfrak{g}_{ss}$, and in particular endows $\mathfrak{r}_{\mathfrak{g}}$ with a $\mathfrak{g}_{ss}$-module structure. The Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathfrak{r}_{\mathfrak{g}} = 0$. It is a classical result of H. Weyl that any (finite-dimensional) semisimple Lie algebra is isomorphic to a direct sum of simple constituents; by definition, a Lie algebra is *simple* if it is not abelian and has no ideals except for the zero one and the entire algebra.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_1

If V is a finite-dimensional vector space, the *general linear Lie algebra* $\mathfrak{gl}(V)$ is defined as the vector space $\operatorname{End} V = V \otimes V^*$ with Lie bracket equal to the commutator of matrix multiplication

$$[x_1, x_2] := x_1 x_2 - x_2 x_1, \tag{1.2}$$

$x_1, x_2 \in \operatorname{End} V$. (More generally, this formula transforms any associative algebra into a Lie algebra.) The radical $\mathfrak{r}_{\mathfrak{gl}(V)}$ equals the center $Z_{\mathfrak{gl}(V)}$ of the Lie algebra $\mathfrak{gl}(V)$, and consists of all scalar multiples of the identity endomorphism of V. Furthermore, the Lie algebra $\mathfrak{gl}(V)$ has a unique Levi component: this is the subalgebra $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ of all endomorphisms having trace zero (for the reader: please make sure you remember why two similar matrices have the same trace). Therefore $\mathfrak{gl}(V)$ is represented canonically as the direct sum $Z_{\mathfrak{gl}(V)} \oplus \mathfrak{sl}(V)$.

It is a well-known fact that the *special linear Lie algebra* $\mathfrak{sl}(V)$ is simple (and thus semisimple) for $\dim V \geq 2$. The cornerstone of the theory of finite-dimensional Lie algebras is the Cartan–Killing classification of simple Lie algebras. It claims that there are four infinite series and five exceptional Lie algebras (up to isomorphism). The Lie algebras $\mathfrak{sl}(V)$ form the series A. More precisely, if $\dim V = \dim V'$, then $\mathfrak{sl}(V)$ and $\mathfrak{sl}(V')$ are clearly isomorphic. Therefore we can define A_i for $i \geq 1$ to be the isomorphism class of $\mathfrak{sl}(V)$ for some vector space V of dimension $i + 1$.

The remaining series, B, C and D, are defined as follows. Let $f : V \times V \to \mathbb{C}$ be a nondegenerate bilinear form on V that is symmetric (i.e., $f(v, w) = f(w, v)$). Then the *orthogonal Lie algebra* $\mathfrak{o}(V)$ is defined by setting

$$\mathfrak{o}(V) := \{x \in \mathfrak{sl}(V) \mid f(x \cdot v, w) = -f(v, x \cdot w)\ \forall v, w \in V\}.$$

A direct computation shows that $\mathfrak{o}(V)$ is a Lie algebra. By definition B_n is the isomorphism class of Lie algebras $\mathfrak{o}(V)$ for $\dim V = 2n + 1$, $n \geq 1$, and D_n is the isomorphism class of $\mathfrak{o}(V)$ for $\dim V = 2n$, $n \geq 2$.

Now suppose that $f : V \times V \to \mathbb{C}$ is a nondegenerate bilinear form on V that is antisymmetric, (i.e., $f(v, w) = -f(w, v)$). Then the dimension of V is necessarily even, and the *symplectic Lie algebra* $\mathfrak{sp}(V)$ is defined as

$$\mathfrak{sp}(V) = \{x \in \mathfrak{sl}(V) \mid f(x \cdot v, w) = -f(v, x \cdot w)\ \forall v, w \in V\}.$$

By C_n one denotes the isomorphism class of $\mathfrak{sp}(V)$ with $\dim V = 2n$ for $n \geq 2$.

In what follows we will refer to V as the natural representation, and we will write $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$ and $\mathfrak{sp}(n)$ under the assumption that V is n-dimensional. We omit the explicit definitions of the five exceptional Lie algebras (the reader should look them up for instance in [HS]), but below we write down their root systems.

1.2 Cartan Subalgebras and Root Systems

A *Cartan subalgebra* of a finite-dimensional Lie algebra $\mathfrak{g}$ is by definition a self-normalizing nilpotent subalgebra. Every finite-dimensional Lie algebra $\mathfrak{g}$ contains a Cartan subalgebra. Moreover it is a fundamental result, [B, Ch.VII, §3, no.2], that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under the group of inner automorphisms of $\mathfrak{g}$.

A *reductive* Lie algebra is by definition a Lie algebra $\mathfrak{g}$ whose radical $\mathfrak{r}_\mathfrak{g}$ equals the center of $\mathfrak{g}$. (The reader should compare this definition with [B, Ch.I, §6, no.4].) As in the case of $\mathfrak{gl}(n)$, any reductive Lie algebra is canonically isomorphic to the direct sum $Z_\mathfrak{g} \oplus \mathfrak{g}_{ss}$, where $Z_\mathfrak{g}$ stands for the center of $\mathfrak{g}$ and $\mathfrak{g}_{ss}$ is identified with the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. In particular, the derived subalgebra of a reductive Lie algebra $\mathfrak{g}$ is the unique Levi component of $\mathfrak{g}$.

An important structural result claims that every Cartan subalgebra $\mathfrak{h}$ of a reductive Lie algebra is abelian and contains only semisimple elements, i.e., elements $x \in \mathfrak{g}$ for which the adjoint operator $\operatorname{ad} x := [x, \cdot]\colon \mathfrak{g} \to \mathfrak{g}$ is semisimple. A subalgebra with this latter property is called *toral*. Moreover the Cartan subalgebras of a reductive Lie algebra are precisely the maximal toral subalgebras [B, Ch.VII, §2, no.4].

Every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contains the center $Z_\mathfrak{g}$, and the projection of any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ into $\mathfrak{g}_{ss}$ is a Cartan subalgebra of $\mathfrak{g}_{ss}$ (see [B, Ch.VII, §2, no.2] for stronger statements). Furthermore, if $\mathfrak{g}$ is reductive, the adjoint operators $\operatorname{ad} h$ with $h \in \mathfrak{h}$ form a system of commuting diagonalizable operators on $\mathfrak{g}$. Since $\mathfrak{g}$ is finite dimensional, these operators are simultaneously diagonalizable, yielding the vector space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha \right),$$

where each α is a linear functional on $\mathfrak{h}$, (i.e., $\alpha \in \mathfrak{h}^*$), and Δ stands for the set of all nonzero functionals appearing in this way (which are called *roots*). Clearly Δ is a finite set, and by definition Δ is the *root system* of $\mathfrak{g}$. The spaces

$$\mathfrak{g}^\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

are one-dimensional and are called the *root spaces* of $\mathfrak{g}$.

It is a fundamental fact that up to isomorphism any reductive Lie algebra $\mathfrak{g}$ is determined by the pair $\{\mathfrak{h}^*, \Delta\}$, $\mathfrak{h}^*$ being considered just as an abstract vector space and Δ being a finite subset of nonzero vectors in $\mathfrak{h}^*$. The pair $\{\operatorname{span} \Delta, \Delta\}$ determines $\mathfrak{g}_{ss}$ up to isomorphism and $\dim Z_\mathfrak{g} = \dim \mathfrak{h}^* / \operatorname{span} \Delta$.

The Cartan–Killing classification states that the following is a complete list of non-isomorphic simple finite-dimensional Lie algebras over $\mathbb{C}$: $\mathfrak{sl}(n)$ for $n \geq 2$, $\mathfrak{o}(m)$ for $m \geq 7$, $\mathfrak{sp}(2n)$ for $n \geq 2$, G_2, F_4, E_6, E_7 and E_8.

In what follows, we give a list of the corresponding root systems. In each case, we specify a finite set Δ of nonzero vectors in the vector space $\mathbb{C}^n$ in terms of the standard basis $\varepsilon_1, \ldots, \varepsilon_n$, which is nothing but a fixed orthonormal basis for $\mathbb{C}^n$ with respect to a (fixed) symmetric nondegenerate bilinear form. The standard coordinates are denoted respectively by $a_1, \ldots, a_n$. In the cases that $\operatorname{span}\Delta \neq \mathbb{C}^n$, we explicitly specify $\operatorname{span}\Delta$ as a subspace of $\mathbb{C}^n$, since $\operatorname{span}\Delta$ equals $\mathfrak{h}^*$ for the corresponding Lie algebra. The root systems of the Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$, $\mathfrak{o}(2n)$ are A_{n-1}, B_n, C_n, D_n, respectively.

$$A_{n-1}:\ n \geq 2,\ \Delta = \left\{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\right\},$$

$$\operatorname{span}\Delta \text{ is the hyperplane with equation } \sum_{i=1}^{n} a_i = 0;$$

$$B_n:\ n \geq 3,\ \Delta = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \mid 1 \leq i < j \leq n\right\};$$

$$C_n:\ n \geq 2,\ \Delta = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_i \mid 1 \leq i < j \leq n\right\};$$

$$D_n:\ n \geq 4,\ \Delta = \left\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\right\};$$

$$G_2:\ n = 3,\ \Delta = \left\{\varepsilon_i - \varepsilon_j,\ \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid 1 \leq i \neq j \neq k \neq i \leq 3\right\},$$

$$\operatorname{span}\Delta \text{ is the hyperplane with equation } \sum_{i=1}^{3} a_i = 0;$$

$$F_4:\ n = 4,\ \Delta = \left\{\pm\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j,\ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \,\middle|\, 1 \leq i < j \leq 4\right\};$$

$$E_6:\ n = 8,\ \Delta = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{k=1}^{5}(-1)^{\nu(k)}\varepsilon_k) \,\middle|\right.$$

$$\left. 1 \neq i < j \leq 5,\ \nu(k) = 1, 2,\ \sum_{k=1}^{5}\nu(k) \in 2\mathbb{Z}\right\},$$

$$\operatorname{span}\Delta \text{ is the 6-dimensional subspace with equations}$$

$$a_6 - a_7 = 0,\ a_7 + a_8 = 0;$$

$$E_7:\ n = 8,\ \Delta = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm(\varepsilon_7 - \varepsilon_8),\ \pm\frac{1}{2}(\varepsilon_7 - \varepsilon_8 + \sum_{k=1}^{6}(-1)^{\nu(k)}\varepsilon_k) \,\middle|\right.$$

$$\left. 1 \leq i < j \leq 6,\ \nu(k) = 1, 2,\ (\sum_{k=1}^{6}\nu(k)) + 1 \in 2\mathbb{Z}\right\},$$

$$\text{span } \Delta \text{ is the hyperplane with equation } a_7 + a_8 = 0;$$

$$E_8 : \ n = 8, \ \Delta = \left\{ \pm\varepsilon_i \pm \varepsilon_j, \ \frac{1}{2}\sum_{k=1}^{8} (-1)^{\nu(k)} \varepsilon_k \ \middle| \ 1 \le i < j \le 8, \ \nu(k) = 1, 2, \ \sum_{k=1}^{8} \nu(k) \in 2\mathbb{Z} \right\}.$$

1.3 Borel Subalgebras and Sets of Simple Roots

Our last topic in this chapter is a quick recollection of Borel subalgebras. A *Borel subalgebra* of a finite-dimensional Lie algebra $\mathfrak{g}$ is by definition a maximal solvable subalgebra. Recall that a subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is solvable, if and only if it contains a chain of subalgebras $\mathfrak{b} = \mathfrak{b}_n \supset \mathfrak{b}_{n-1} \supset \cdots \supset \mathfrak{b}_0 = 0$ such that each $\mathfrak{b}_{k-1}$ is an ideal in $\mathfrak{b}_k$ for $k = 1, \ldots, n$ and the quotient $\mathfrak{b}_k/\mathfrak{b}_{k-1}$ is abelian. In particular, a Borel subalgebra is a subalgebra of $\mathfrak{g}$ that is maximal with respect to this property. Moreover every Borel subalgebra of $\mathfrak{g}$ contains the radical $\mathfrak{r}_\mathfrak{g}$. (Prove!) It is also an exercise to show that if $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a surjective homomorphism of finite-dimensional Lie algebras, the image of every Borel subalgebra of $\mathfrak{g}$ is a Borel subalgebra of $\mathfrak{g}'$.

If $\mathfrak{g}$ is reductive, then any Borel subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{h} \supsetneq \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra (in this case, maximal toral subalgebra), and $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is an ideal in $\mathfrak{b}$ containing only nilpotent elements. It is also well known that as a subalgebra of $\mathfrak{g}$, $\mathfrak{n}$ is maximal with respect to this latter property. Furthermore, $\mathfrak{b}$ equals the normalizer $N_\mathfrak{g}(\mathfrak{n})$ of $\mathfrak{n}$ in $\mathfrak{g}$, and $\mathfrak{n}$ determines $\mathfrak{b}$ uniquely. In addition, one can show that the correspondence $\mathfrak{b} \to \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ is a bijection between Borel subalgebras of $\mathfrak{g}$ and maximal subalgebras of nilpotent elements in $\mathfrak{g}$. On the other hand, a Borel subalgebra $\mathfrak{b}$ of a nonabelian reductive Lie algebra $\mathfrak{g}$ contains infinitely many Cartan subalgebras, which are characterized by the fact that they are subalgebras $\mathfrak{h}'$ of $\mathfrak{b}$ such that the sum of vector spaces $\mathfrak{h}' \oplus \mathfrak{n}$ equals $\mathfrak{b}$.

Next we present the most common characterization of Borel subalgebras. For the rest of this chapter, we fix a reductive Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. It is an exercise to prove that any subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ that contains a Cartan subalgebra $\mathfrak{h}$ is a *root subalgebra* of $\mathfrak{g}$, that is it decomposes as $\mathfrak{k} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_\mathfrak{k}} \mathfrak{g}^\alpha)$ for some subset $\Delta_\mathfrak{k} \subset \Delta$. In particular, any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{b} \supset \mathfrak{h}$ is characterized by a (finite) subset $\Delta_\mathfrak{b} \subset \Delta$. It is a well-known result that all such subsets $\Delta_\mathfrak{b}$ are precisely the subsets of Δ that satisfy the following two conditions:

(B1) $\Delta_\mathfrak{b}$ is additive within Δ, i.e., $\alpha \in \Delta_\mathfrak{b}$, $\beta \in \Delta_\mathfrak{b}$, and $\alpha + \beta \in \Delta$ imply $\alpha + \beta \in \Delta_\mathfrak{b}$;

(B2) for any $\alpha \in \Delta$, exactly one of the following possibilities holds: $\alpha \in \Delta_\mathfrak{b}$ or $-\alpha \in \Delta_\mathfrak{b}$.

Furthermore, such a set $\Delta_{\mathfrak{b}}$ is determined uniquely by the *simple roots* $\sum_{\mathfrak{b}} \subset \Delta_{\mathfrak{b}}$ of $\mathfrak{b}$. These are by definition the indecomposable elements in $\Delta_{\mathfrak{b}}$, i.e., the elements in $\Delta_{\mathfrak{b}}$ that are not decomposable as the sum of two elements of $\Delta_{\mathfrak{b}}$.

The following is a deep classical result [B, Ch.VIII, §3, no.3]. Let $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ be two Cartan subalgebras (allowing for the possibility $\mathfrak{h}_1 = \mathfrak{h}_2$) and let $\mathfrak{b}_1 \supset \mathfrak{h}_1$ and $\mathfrak{b}_2 \supset \mathfrak{h}_2$ be two Borel subalgebras of $\mathfrak{g}$. Then there exists an inner automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$ and $\varphi(\mathfrak{b}_1) = \varphi(\mathfrak{b}_2)$. In addition, such an automorphism is automatically compatible with the simple roots, i.e., $\varphi(\sum_{\mathfrak{b}_1}) = \sum_{\mathfrak{b}_2}$.

This shows that instead of characterizing $\mathfrak{g}$ by the pair $\{\mathfrak{h}^*, \Delta\}$, one can try to characterize $\mathfrak{g}$ by the pair $\{\mathfrak{h}^*, \sum_{\mathfrak{b}}\}$ for some Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Abstractly, the pair $\{\mathfrak{h}^*, \sum_{\mathfrak{b}}\}$ is nothing but a vector space and a set of linearly independent vectors in it which certainly does not characterize $\mathfrak{g}$. However, $\mathfrak{h}^*$ is endowed with a scalar product defined via a (fixed) symmetric nondegenerate bilinear form $(\cdot\,,\,\cdot)$, which is the bilinear form on $\mathbb{C}^n$ from our explicit descriptions of the root systems. This form was not used above, but now we can use it to recover $\mathfrak{g}$ via $\{\mathfrak{h}^*, \sum_{\mathfrak{b}}\}$. Indeed it is an important classical fact that a reductive Lie algebra is characterized up to isomorphism by any pair $\{\mathfrak{h}^*, \sum_{\mathfrak{b}}\}$ together with the matrix of all scalar products (σ_i, σ_j) of simple roots $\sigma_i, \sigma_j \in \sum_{\mathfrak{b}}$. Traditionally, the latter information is encoded in the Dynkin diagrams of the simple Lie algebras (to the reader: please recall the definition of a Dynkin diagram), and $\dim Z_{\mathfrak{g}} = \dim \mathfrak{h}^* / \operatorname{span} \sum_{\mathfrak{b}}$.

In the explicit realizations of the root systems given above, it is enough to specify $\sum_{\mathfrak{b}}$ as a subset of Δ, since the scalar products (σ_i, σ_j) are determined through the standard scalar product on $\mathbb{C}^n$. Here is a specific choice of simple roots for each isomorphism class of simple finite-dimensional Lie algebras:

$$
\begin{aligned}
A_{n-1} &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3, \ldots,\ \varepsilon_{n-1} - \varepsilon_n;\\
B_n &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3, \ldots,\ \varepsilon_{n-1} - \varepsilon_n,\ \varepsilon_n;\\
C_n &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3, \ldots,\ \varepsilon_{n-1} - \varepsilon_n,\ 2\varepsilon_n;\\
D_n &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3, \ldots,\ \varepsilon_{n-1} - \varepsilon_n,\ \varepsilon_{n-1} + \varepsilon_n;\\
G_2 &: \ \varepsilon_1 - \varepsilon_2,\ -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3;\\
F_4 &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3,\ \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4);\\
E_6 &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_5,\ \varepsilon_4 + \varepsilon_5,\\
&\qquad \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8);\\
E_7 &: \ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_5,\ \varepsilon_5 - \varepsilon_6,\ \varepsilon_5 + \varepsilon_6,\\
&\qquad \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8)
\end{aligned}
$$

$$E_8 : \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_5,\ \varepsilon_5 - \varepsilon_6,\ \varepsilon_6 - \varepsilon_7,$$
$$\varepsilon_6 + \varepsilon_7,\ \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 + \varepsilon_8).$$

Note that our descriptions of the root systems F_4, E_6, E_7 and E_8, along with their simple roots, are equivalent but not identical to those of [B, Ch.VI].

Next we recall that a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a *parabolic subalgebra* if it contains a Borel subalgebra. A parabolic subalgebra that is not a Borel subalgebra will contain more than one Borel subalgebra even for fixed $\mathfrak{h}$. Any parabolic subalgebra $\mathfrak{p}$ is a root subalgebra of $\mathfrak{g}$ with respect to any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{p}$. Moreover, for a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, the parabolic subalgebras containing $\mathfrak{b}$ are in 1-1 correspondence with subsets of the set $\sum_{\mathfrak{b}}$ of simple roots of $\mathfrak{b}$: given any subset $S \subset \sum_{\mathfrak{b}}$, we define $\mathfrak{p}_S$ to be the parabolic subalgebra generated by root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \sum_{\mathfrak{b}} \cup (-S)$.

Recall that if $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra, then the *Weyl group* W is by definition the subgroup of the orthogonal group $O(\mathfrak{h}^*)$ that is generated by the reflections s_α with $\alpha \in \Delta$, where $s_\alpha(v) := v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$ for $v \in \mathfrak{h}^*$. Since all Cartan subalgebras of $\mathfrak{g}$ are conjugate by the group of inner automorphisms, the Weyl groups defined for different Cartan subalgebras are isomorphic. In fact, they are canonically isomorphic. It is a classical result that the Weyl group W of $\mathfrak{g}$ acts simply transitively on the set of all sets of simple roots for the root system Δ of $\mathfrak{g}$. In the next chapter we will see how things can differ for Lie superalgebras.

Chapter 2
Finite-Dimensional Lie Superalgebras

The theory of finite-dimensional Lie superalgebras is a natural generalization of the theory of finite-dimensional Lie algebras, and it is far more complicated and less developed. Nevertheless the simple finite-dimensional Lie superalgebras have now been classified for more than 40 years, and their classification is a fundamental result, see [K2]. In this chapter we will go over very briefly the same topics for Lie superalgebras as we did in Chap. 1 for Lie algebras.

2.1 Lie Superalgebras

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with an even (i.e., grading preserving) bilinear map

$$[\cdot\,,\,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the *supercommutator*, with the properties

(1) $[x_1, x_2] = -(-1)^{\bar{x}_1\bar{x}_2}[x_2, x_1]$ (super anticommutativity),
(2) $(-1)^{\bar{x}_1\bar{x}_3}[[x_1, x_2], x_3] + (-1)^{\bar{x}_2\bar{x}_3}[[x_3, x_1], x_2] + (-1)^{\bar{x}_1\bar{x}_2}[[x_2, x_3], x_1] = 0$
("Jacobi identity with signs").

Here x_i are $\mathbb{Z}_2$-homogeneous elements of $\mathfrak{g}$, and $\bar{x}_i \in \mathbb{Z}_2 = \{0, 1\}$ denotes the parity of an element x_i (i.e., $x_i \in \mathfrak{g}_{\bar{x}_i}$). This definition is completely analogous to that of a Lie algebra but has the remarkable additional signs. Note that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. (Prove it directly from the definition!)

Many notions such as nilpotency, solvability, simplicity, etc., carry over immediately to Lie superalgebras, with the additional assumption that all vector spaces, subalgebras and ideals are assumed to have a $\mathbb{Z}_2$-grading. The reader is encouraged to recover these notions on his/her own, or to find the relevant definitions in

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_2

[ChW, K2, Mu, Sch1]. The latter, together with [P3], are our main references for this chapter. However the structure theory for Lie superalgebras is less complete than the one for Lie algebras, and below we will give an introduction to this theory.

First let's consider some examples.

Example 2.1 Let $M_{r,s}$ denote the space of $r \times s$-matrices. We can define a Lie superalgebra structure on the space $M_{m+n,m+n}$, with the $\mathbb{Z}_2$-grading

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in M_{m,m},\ D \in M_{n,n} \right\},$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B \in M_{m,n},\ C \in M_{n,m} \right\},$$

by letting the supercommutator be given by

$$[x_1, x_2] := x_1 x_2 - (-1)^{\bar{x}_1 \bar{x}_2} x_1 x_2$$

on $\mathbb{Z}_2$-homogeneous elements $x_1, x_2 \in \mathfrak{g}$, and then by extending it linearly to $M_{m+n,m+n}$. This defines the *general linear Lie superalgebra* $\mathfrak{gl}(m|n)$. ◯

The $\mathbb{Z}_2$-grading on $\mathfrak{gl}(m|n)$ leads to additional sign changes in some common linear algebra formulas. The *supertranspose* of a matrix $x_1 \in \mathfrak{gl}(m|n)$ is defined to be

$$(x_1)^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T := \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix},$$

where A^t denotes the usual matrix transpose. The *supertrace* of $x_1 \in \mathfrak{gl}(m|n)$ is given by

$$\operatorname{str} x_1 = \operatorname{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \operatorname{tr} A - \operatorname{tr} D.$$

Example 2.2 The *special linear Lie superalgebra* $\mathfrak{sl}(m|n)$ is the subsuperalgebra of $\mathfrak{gl}(m|n)$ composed of all matrices that have the supertrace equal to zero. (To the reader: check that $\mathfrak{sl}(m|n)$ is an ideal in $\mathfrak{gl}(m|n)$ and in particular a Lie subsuperalgebra of $\mathfrak{gl}(m|n)$.) When $m \neq n$ and $m, n \geq 1$, the Lie superalgebra $\mathfrak{sl}(m|n)$ is simple (check this!). The Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(n|n)$ is not simple, since it has a non-trivial (abelian) ideal spanned by the identity matrix $I = \operatorname{span}\{\mathrm{id}\}$. When $n = 1$, the quotient $\mathfrak{g}/I$ is abelian, and moreover $[\mathfrak{g}, \mathfrak{g}] = I = Z_{\mathfrak{g}}$ implies that $\mathfrak{g} = \mathfrak{sl}(1|1)$ is a nilpotent Lie superalgebra. When $n \geq 2$, the quotient $\mathfrak{sl}(n|n)/I$ is a simple Lie superalgebra denoted $\mathfrak{psl}(n|n)$. ◯

For any finite-dimensional Lie superalgebra $\mathfrak{g}$, the exact sequence (1.1) makes perfect sense, where $\mathfrak{r}_\mathfrak{g}$ is now defined as the maximal solvable ideal in the Lie superalgebra $\mathfrak{g}$. The quotient $\mathfrak{g}_{ss} = \mathfrak{g}/\mathfrak{r}_\mathfrak{g}$ is by definition the *semisimple part* of $\mathfrak{g}$, and is *semisimple*, i.e., it has no nontrivial solvable ideals. An immediate difference from the Lie algebra case is that the sequence (1.1) does not always split for Lie superalgebras. We will give such an example below.

A further drastic difference is that $\mathfrak{g}_{ss}$ is not necessarily isomorphic to a direct sum of *simple* Lie superalgebras, i.e., Lie superalgebras that have no nontrivial proper ideals. Here is an elementary example of a semisimple Lie superalgebra that does not decompose into a direct sum of simple Lie superalgebras.

Example 2.3 Consider the Lie superalgebra

$$\mathfrak{g} = (\mathfrak{sl}(2) \otimes \operatorname{span}\{1, \xi\}) \dot{+} \operatorname{span}\{\frac{d}{d\xi}\},$$

where ξ is an odd variable satisfying $\xi^2 = 0$, and $\frac{d}{d\xi}$ is the corresponding odd superderivation: $\frac{d}{d\xi}(\xi) = 1$, $\xi \frac{d}{d\xi} + \frac{d}{d\xi}\xi = 1$. Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2)$ and $\mathfrak{g}_{\bar{1}} = (\mathfrak{sl}(2) \otimes \operatorname{span}\{\xi\}) \dot{+} \operatorname{span}\{\frac{d}{d\xi}\}$.

The restriction of the supercommutator $[\cdot\,,\,\cdot]$ of $\mathfrak{g}$ to $\mathfrak{sl}(2)$ is just the usual commutator of $\mathfrak{sl}(2)$. The elements ξ and $\frac{d}{d\xi}$ commute with $\mathfrak{sl}(2)$, so for any $x_1, x_2 \in \mathfrak{sl}(2)$ we have the following relations:

$$\left[x_1, \tfrac{d}{d\xi}\right] = 0, \qquad [x_1, x_2 \otimes \xi] = [x_1, x_2] \otimes \xi,$$

$$\left[x_1 \otimes \xi, \tfrac{d}{d\xi}\right] = x_1, \qquad [x_1 \otimes \xi, x_2 \otimes \xi] = [x_1, x_2] \otimes \xi^2 = 0,$$

$$\left[\tfrac{d}{d\xi}, \tfrac{d}{d\xi}\right] = 0.$$

As our notation suggests, $\tilde{\mathfrak{g}} := \mathfrak{sl}(2) \otimes \operatorname{span}\{1, \xi\}$ is an ideal in $\mathfrak{g}$. However, $\tilde{\mathfrak{g}}_{\bar{0}} = \mathfrak{sl}(2)$ and thus $\tilde{\mathfrak{g}}_{\bar{0}}$ is not a solvable Lie superalgebra. The subsuperalgebra $\tilde{\mathfrak{g}}_{\bar{1}}$ is an abelian (odd) subsuperalgebra of $\mathfrak{g}$, but it contains no nontrivial ideal of $\mathfrak{g}$. Moreover the radical of $\mathfrak{g}$ is easily checked to be zero, and thus $\mathfrak{g} = \mathfrak{g}_{ss}$, i.e., $\mathfrak{g}$ is semisimple.

On the other hand, $\mathfrak{g}$ is not a direct sum of simple Lie superalgebras. Indeed, if $\mathfrak{g}$ were the direct sum of two ideals $\hat{\mathfrak{g}}$ and $\hat{\hat{\mathfrak{g}}}$, we would have $\hat{\mathfrak{g}}_{\bar{0}} = 0$ or $\hat{\hat{\mathfrak{g}}}_{\bar{0}} = 0$ as $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2)$. Assume $\hat{\mathfrak{g}}_{\bar{0}} = 0$. Then $\hat{\mathfrak{g}}$ would be an abelian odd Lie subsuperalgebra of $\mathfrak{g}$, and thus a Lie subsuperalgebra of $\operatorname{span}\{\frac{d}{d\xi}\}$ or of $\mathfrak{sl}(2) \otimes \operatorname{span}\{\xi\}$. However no such abelian subsuperalgebra is an ideal in $\mathfrak{g}$, hence $\mathfrak{g} \not\simeq \hat{\mathfrak{g}} \oplus \hat{\hat{\mathfrak{g}}}$. ◯

This example shows that semisimple Lie superalgebras are much more complicated objects than direct sums of simple Lie superalgebras. The Kac–Cheng Theorem provides a complete and almost explicit description of all finite-dimensional semisimple Lie superalgebras. We will state this theorem below, but first we recall some basic facts about simple Lie superalgebras.

2.2 Classical Lie Superalgebras

There are several families of Lie superalgebras whose definitions are completely analogous to those of the classical Lie algebras, that is these Lie superalgebras consist of endomorphisms of a vector space V that preserve some (additional) natural structure on V, (for instance, a bilinear form), except that now V has a $\mathbb{Z}_2$-grading, $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

2.2.1 Definitions

The General Linear Lie Superalgebra $\mathfrak{gl}(V)$

The vector space $\operatorname{End} V = V \otimes V^*$ is $\mathbb{Z}_2$-graded with grading

$$(V \otimes V^*)_{\bar{0}} = ((V_{\bar{0}} \otimes V_{\bar{0}}^*) \oplus (V_{\bar{1}} \otimes V_{\bar{1}}^*)) \text{ and } (V \otimes V^*)_{\bar{1}} = ((V_{\bar{0}} \otimes V_{\bar{1}}^*) \oplus (V_{\bar{1}} \otimes V_{\bar{0}}^*)),$$

and moreover has a natural Lie superalgebra structure with supercommutator

$$[x_1, x_2] = x_1 x_2 - (-1)^{\tilde{x}_1 \tilde{x}_2} x_2 x_1 \tag{2.1}$$

on $\mathbb{Z}_2$-homogeneous endomorphisms $x_1, x_2 \in \operatorname{End} V$. (The product of endomorphisms in the right-hand side of (2.1) is of course a composition of endomorphisms.) This is the general linear Lie superalgebra $\mathfrak{gl}(V)$. The even part of $\mathfrak{gl}(V)$ is $\mathfrak{gl}(V)_{\bar{0}} = \mathfrak{gl}(V_{\bar{0}}) \oplus \mathfrak{gl}(V_{\bar{1}})$. With respect to a suitable basis, $\mathfrak{gl}(V)$ consists of block matrices of the form given in Example 2.1, where $m = \dim V_{\bar{0}}$ and $n = \dim V_{\bar{1}}$, and there is an obvious isomorphism of Lie superalgebras $\mathfrak{gl}(V) \cong \mathfrak{gl}(m|n)$.

The special linear Lie superalgebra $\mathfrak{sl}(V)$ is the Lie subsuperalgebra of $\mathfrak{gl}(V)$ determined by the equation $\operatorname{str} x = 0$. In particular, $\mathfrak{sl}(V) \cong \mathfrak{sl}(m|n)$ and $\mathfrak{sl}(V)_{\bar{0}} \cong \mathfrak{sl}(V_{\bar{0}}) \oplus \mathfrak{sl}(V_{\bar{1}}) \oplus \mathbb{C}$. In addition, if $n = \dim V_{\bar{0}} = \dim V_{\bar{1}}$, then $\mathrm{id}_V \in \mathfrak{sl}(V)$, and thus the Lie superalgebra $\mathfrak{psl}(V) := \mathfrak{sl}(V)/\operatorname{span}\{\mathrm{id}_V\}$ is well defined with $\mathfrak{psl}(V)_{\bar{0}} \cong \mathfrak{sl}(V_{\bar{0}}) \oplus \mathfrak{sl}(V_{\bar{1}})$.

This latter construction shows in particular that the Levi–Malcev Theorem does not hold for Lie superalgebras. Indeed, consider the exact sequence

$$0 \to \mathbb{C} \to \mathfrak{sl}(V) \to \mathfrak{psl}(V) \to 0,$$

where $\dim V_{\bar{0}} = \dim V_{\bar{1}}$. Then this is nothing but sequence (1.1) for $\mathfrak{g} = \mathfrak{sl}(V)$, and it is an exercise (please do it!) to show that this sequence does not split.

The isomorphism class of $\mathfrak{sl}(V)$ is denoted by $A_{m-1|n-1}$ for $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$, $m \neq n$, while $A_{n-1|n-1}$ is the isomorphism class of $\mathfrak{psl}(V)$ with $\dim V_{\bar{0}} = \dim V_{\bar{1}} = n$.

The Orthosymplectic Lie Superalgebra $\mathfrak{osp}(V)$

Assume now that V is equipped with a nondegenerate even supersymmetric bilinear form, i.e., with a parity preserving nondegenerate pairing $f : V \times V \to \mathbb{C}$ such that $f(v, w) = (-1)^{\bar{v}\bar{w}} f(w, v)$ for $\mathbb{Z}_2$-homogeneous vectors $v, w \in V$. The Lie superalgebra $\mathfrak{osp}(V) \subset \mathfrak{gl}(V)$ is defined as follows. For $\bar{i} \in \mathbb{Z}_2$, let

$$\mathfrak{osp}(V)_{\bar{i}} := \operatorname{span}\left\{ x \in \mathfrak{gl}(V)_{\bar{i}} \;\middle|\; \begin{array}{c} f(x \cdot v, w) = -(-1)^{\bar{x}\bar{v}} f(v, x \cdot w), \\ \forall v \in V_{\bar{v}}, \forall w \in V \end{array} \right\}.$$

Then $\mathfrak{osp}(V) = \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}$ is called the *orthosymplectic Lie superalgebra*, and as the name suggests, $\mathfrak{osp}(V)$ is a crossbreed between the orthogonal and the symplectic Lie algebras. The reader should check that $\mathfrak{osp}(V)$ is indeed a Lie subsuperalgebra of $\mathfrak{sl}(V)$ and that $\mathfrak{osp}(V)_{\bar{0}} = \mathfrak{o}(V_{\bar{0}}) \oplus \mathfrak{sp}(V_{\bar{1}})$. It follows that the dimension of $V_{\bar{1}}$ is necessarily even.

The orthosymplectic Lie superalgebras can be separated into three families according to the dimension of $V_{\bar{0}}$. We denote by $B_{m|n}$ the isomorphism class of $\mathfrak{osp}(V)$ with $\dim V_{\bar{0}} = 2m + 1 \geq 1$, $\dim V_{\bar{1}} = 2n \geq 0$, $(m, n) \neq (0, 0)$. Similarly, $C_{1|n}$ stands for the isomorphism class of $\mathfrak{osp}(V)$ with $\dim V_{\bar{0}} = 2$, $\dim V_{\bar{1}} = 2n \geq 2$, and $D_{m|n}$ is the isomorphism class of $\mathfrak{osp}(V)$ with $\dim V_{\bar{0}} = 2m \geq 0$, $\dim V_{\bar{1}} = 2n \geq 0$, $m \neq 1$, $(m, n) \neq (0, 2)$.

If $\mathfrak{g} = \mathfrak{osp}(V)$ is of type $B_{m|n}$, then a basis for V can be chosen so that $\mathfrak{g}$ has the block matrix form

$$\left(\begin{array}{ccc|cc} 0 & p & q & r & s \\ -q^t & A & B & W & X \\ -p^t & C & -A^t & Y & Z \\ \hline -s^t & -Z^t & -X^t & D & E \\ r^t & Y^t & W^t & F & -D^t \end{array}\right), \tag{2.2}$$

where A is an $(m \times m)$-matrix, B and C are skew-symmetric $(m \times m)$-matrices, D is an $(n \times n)$-matrix, E and F are symmetric $(n \times n)$-matrices, W, X, Y, Z are $(m \times n)$-matrices, p and q are $(1 \times m)$-matrices, and r and s are $(1 \times n)$-matrices.

If $\mathfrak{g} = \mathfrak{osp}(V)$ is of type $C_{1|n}$ or $D_{m|n}$, then with respect to a suitable basis $\mathfrak{g}$ consists of block matrices of the form (2.2) except with the first row and first column removed (where for type $C_{1|n}$ we take $m = 1$).

The Lie superalgebra $\mathfrak{osp}(2m + 1|2n)$ is defined to be the Lie subsuperalgebra of $\mathfrak{gl}(2m + 1|2n)$ consisting of all matrices of the form (2.2), and the Lie superalgebra $\mathfrak{osp}(2m|2n)$ is the Lie subsuperalgebra of $\mathfrak{gl}(2m|2n)$ consisting of all matrices given by the form (2.2) with the first row and first column removed.

The Lie Superalgebra $\mathbf{p}(V)$

Next let V be equipped with a nondegenerate odd supersymmetric bilinear form, i.e., with a nondegenerate pairing $f : V \times V \to \mathbb{C}$ satisfying

$$f(V_{\bar{0}}, V_{\bar{0}}) = 0,\ f(V_{\bar{1}}, V_{\bar{1}}) = 0, \text{ and } f(v, w) = f(w, v) \text{ for all } v \in V_{\bar{0}},\ w \in V_{\bar{1}}.$$

Note that the existence of such a form f implies that $\dim V_{\bar{0}} = \dim V_{\bar{1}}$.

The linear transformations of V that are invariant with respect to f form a Lie subsuperalgebra $\mathbf{p}(V) \subset \mathfrak{gl}(V)$. Explicitly, this means that we have

$$\mathbf{p}(V)_{\bar{i}} = \operatorname{span}\left\{ x \in \mathfrak{gl}(V)_{\bar{i}} \;\middle|\; \begin{array}{c} f(x \cdot v, w) = -(-1)^{\bar{x}\bar{v}} f(v, x \cdot w), \\ \forall v \in V_{\bar{v}}, \forall w \in V \end{array} \right\}$$

for each $\bar{i} \in \mathbb{Z}_2$. The reader should check that $\mathbf{p}(V) = \mathbf{p}(V)_{\bar{0}} \oplus \mathbf{p}(V)_{\bar{1}}$ is a Lie superalgebra, and that $\mathbf{p}(V)_{\bar{0}} = \mathfrak{gl}(V_{\bar{0}})$.

With respect to a suitable basis, the Lie superalgebra $\mathbf{p}(V)$ consists of block matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array}\right),$$

where B is symmetric and C is skew-symmetric. Imposing the additional condition that $\operatorname{tr} A = 0$ defines the Lie superalgebra $\mathfrak{sp}(V)$, which is simple when $\dim V_{\bar{0}} \geq 3$.

The Lie Superalgebra $\mathfrak{q}(V)$

Finally, V may be equipped with a Π-symmetry, that is with an odd (i.e., parity changing) linear map $\Pi : V \to V$ satisfying $\Pi^2 = \mathrm{id}_V$. For each $\bar{i} \in \mathbb{Z}_2$, we set

$$\mathfrak{q}(V)_{\bar{i}} := \operatorname{span}\{x \in \mathfrak{gl}(V)_{\bar{i}} \mid x \cdot (\Pi v) = \Pi\left((-1)^{\bar{x}} x \cdot v\right)\ \forall v \in V\}.$$

A straightforward computation shows that $\mathfrak{q}(V) = \mathfrak{q}(V)_{\bar{0}} \oplus \mathfrak{q}(V)_{\bar{1}}$ is a Lie subsuperalgebra of $\mathfrak{gl}(V)$ and that $\mathfrak{q}(V)_{\bar{0}} = \mathfrak{gl}(V_{\bar{0}})$. Furthermore, it is not difficult to verify that $\mathfrak{q}(V)_{\bar{1}} \simeq V_{\bar{0}} \otimes V_{\bar{1}}^*$, and since $V_{\bar{0}}$ and $V_{\bar{1}}$ are swapped by Π, we can identify $\mathfrak{q}(V)_{\bar{1}}$ with $\Pi(V_{\bar{0}} \otimes V_{\bar{0}}^*)$. For $x = x_0 + x_1 \in \mathfrak{q}(V)$ with $x_0 \in \mathfrak{q}(V)_{\bar{0}}$, $x_1 \in \mathfrak{q}(V)_{\bar{1}}$, we call $\operatorname{otr}(x) := \operatorname{tr}(\Pi(x_1))$ the *odd trace* of x, and note that it makes sense to impose the equation $\operatorname{tr} x_1 = 0$. It is a key observation that this equation determines a well-defined Lie subsuperalgebra of $\mathfrak{q}(V)$, which we denote by $\mathfrak{sq}(V)$.

An easy way to check that $\mathfrak{sq}(V)$ is indeed a Lie superalgebra is to fix a homogeneous basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ in $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $n = \dim V_{\bar{0}} = \dim V_{\bar{1}}$, such that $w_i = \Pi(v_i)$. Then one can verify that in this basis $\mathfrak{q}(V)$ consists

of block matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right),$$

where each block has size $n \times n$. In this way the condition $\mathrm{otr}(x) = 0$ becomes $\mathrm{tr}\, B = 0$. Moreover, the observation that $\mathfrak{sq}(V)$ is a Lie subsuperalgebra reduces to a trivial calculation showing that the supercommutator preserves the condition $\mathrm{tr}\, B = 0$. Finally, note that $\mathrm{id}_V \in \mathfrak{sq}(V)$, and therefore the Lie superalgebra $\mathfrak{psq}(V) := \mathfrak{sq}(V)/\operatorname{span}\{\mathrm{id}_V\}$ is also well defined.

Classical Lie Superalgebras

The Lie superalgebras of types $\mathfrak{gl}$, $\mathfrak{osp}$, $\mathbf{p}$ and $\mathfrak{q}$, together with their relatives $\mathfrak{sl}$, $\mathfrak{psl}$, $\mathfrak{sp}$, $\mathfrak{pq}$, $\mathfrak{sq}$ and $\mathfrak{psq}$, are the *classical Lie superalgebras*. The Lie superalgebra $\mathfrak{pq}$, which we did not yet introduce, is defined in the same way as $\mathfrak{psq}$ with $\mathfrak{sq}(V)$ replaced by $\mathfrak{q}(V)$. As we saw, the classical Lie superalgebras are remarkably simple generalizations of the classical Lie algebra $\mathfrak{gl}$, $\mathfrak{o}$, and $\mathfrak{sp}$. The main difference is that for $\mathbb{Z}_2$-graded spaces, the notion of a symmetric even bilinear form generalizes both notions of a symmetric and antisymmetric form from the non-$\mathbb{Z}_2$-graded set-up, and that in addition, there are two more natural structures on a $\mathbb{Z}_2$-graded space V: a nondegenerate odd bilinear form and a Π-symmetry. These latter structures have no analog in the non-Z_2-graded case, and for this reason the Lie superalgebras of types $\mathbf{p}$ and $\mathfrak{q}$ are sometimes called *strange*.

In what follows we will sometimes denote the classical Lie superalgebras by $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, etc., assuming that $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$. In the cases of $\mathbf{p}$, $\mathfrak{q}$, $\mathfrak{sp}$, $\mathfrak{sq}$, $\mathfrak{pq}$, $\mathfrak{psq}$, we write simply $\mathbf{p}(n)$, $\mathfrak{q}(n)$ etc. for $\dim V_{\bar{0}} = \dim V_{\bar{1}} = n$.

2.2.2 Invariant Bilinear Forms

The Killing form of a Lie algebra is an important tool used in the proof of the Cartan–Killing classification of finite-dimensional simple Lie algebras. The Killing form allows one to reduce the problem of classifying simple Lie algebras to a geometric problem of classifying irreducible root systems via Dynkin diagrams. The definition of the Killing form naturally generalizes to the Lie superalgebra setting, namely one sets

$$\kappa(x, y) := \mathrm{str}((\mathrm{ad}\, x) \circ (\mathrm{ad}\, y)),$$

for x and y in a Lie superalgebra $\mathfrak{g}$, and the Killing form is an invariant supersymmetric bilinear form on $\mathfrak{g}$. However, contrary to the Lie algebra case the Killing form

of a simple Lie superalgebra can be degenerate, and hence equal to zero! In fact, the Killing form is identically zero for the Lie superalgebras $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(2n+2|2n)$, $\mathfrak{sp}(n)$ and $\mathfrak{sq}(n)$ [K2].

On the other hand if $\mathfrak{g}$ is a simple classical Lie superalgebra that is not of type $\mathfrak{sp}$, then $\mathfrak{g}$ admits a nondegenerate supersymmetric invariant bilinear form $(\cdot,\cdot)$, which is even for types A, B, C and D, (i.e., $(V_{\bar{0}}, V_{\bar{1}}) = 0$ and $(V_{\bar{1}}, V_{\bar{0}}) = 0$), and is odd for $\mathfrak{sq}$, (i.e., $(V_{\bar{0}}, V_{\bar{0}}) = 0$ and $(V_{\bar{1}}, V_{\bar{1}}) = 0$) [K2]. It is an exercise to show that any two such forms on a simple Lie superalgebra are proportional.

To construct such a form for the cases A, B, C, D, we first define a nondegenerate supersymmetric invariant even bilinear form on $\mathfrak{gl}(m|n)$ by setting $(x, y) := \mathrm{str}(xy)$ for $x, y \in \mathfrak{gl}(m|n)$. The form remains nondegenerate when restricted to the Lie subsuperalgebras $\mathfrak{sl}(m|n)$ for $m \neq n$ and $\mathfrak{osp}(m|2n)$ for arbitrary m and n. The identity matrix I is in the kernel of this form for $\mathfrak{sl}(n|n)$. This enables us to define a nondegenerate supersymmetric invariant even bilinear form on $\mathfrak{psl}(n|n) = \mathfrak{sl}(n|n)/I$ by setting $(x + I, y + I) := \mathrm{str}(xy)$ for $x + I,\ y + I \in \mathfrak{psl}(n|n)$, where $x, y \in \mathfrak{sl}(n|n)$ are representatives of the cosets $x + I$, $y + I$, respectively (prove!).

For $\mathfrak{q}(n)$ the same idea leads to setting $(x, y) := \mathrm{otr}(xy)$ for $x, y \in \mathfrak{q}(n)$. Since otr is an odd linear map, we obtain an odd bilinear form which is easily checked to be invariant and nondegenerate. When restricted to $\mathfrak{sq}(n)$ this form becomes degenerate with kernel equal to the span of the identity matrix, which allows us to define a nondegenerate odd invariant bilinear form on $\mathfrak{psq}(n)$ by setting $(x + I, y + I) := \mathrm{otr}(xy)$ for $x + I,\ y + I \in \mathfrak{psq}(n)$.

The Lie superalgebra $\mathfrak{sp}$ does not admit a nondegenerate even or odd invariant bilinear form. (Prove!)

2.3 Simple Lie Superalgebras

In addition to the classical Lie superalgebras and apart from Lie algebras, there are two exceptional finite-dimensional simple Lie superalgebras which we denote by $G_{2|1}$ and $F_{3|1}$ (the common notations are $G(3)$ and $F(4)$, respectively), as well as a 1-parameter family of exceptional simple Lie superalgebras $D(2, 1; \alpha)$. For the definitions of these exceptional Lie superalgebras we refer the reader to [K2]; below we only present their root systems. Moreover the classical and exceptional simple Lie superalgebras are precisely the simple Lie superalgebras $\mathfrak{g}$ whose even part $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra.[1]

It is a remarkable fact that there is another way to construct simple finite-dimensional Lie superalgebras that have no analogs in the Lie algebra case, or

[1] In order to avoid possible confusion, we point out that in [Mu], any simple Lie superalgebra with a reductive even part is called *classical simple*, while we reserve the term *classical* for the Lie superalgebras discussed in Sect. 2.2. These do not include the Lie superalgebras of exceptional type.

more precisely, whose Lie algebra analogs are infinite dimensional. Indeed, let x be a formal (even) variable. Then the derivations $\operatorname{der}\mathbb{C}[x]$ of the polynomial algebra $\mathbb{C}[x]$ certainly form a Lie algebra (the Witt algebra), but this Lie algebra is infinite dimensional and thus it cannot appear in the Cartan–Killing classification. In contrast, if ξ is an odd variable (in particular $\xi^2 = 0$), then the Lie superalgebra with elements $\alpha\xi\frac{d}{d\xi} + \beta\frac{d}{d\xi}$, $\alpha, \beta \in \mathbb{C}$, and relations $[\xi\frac{d}{d\xi}, \xi\frac{d}{d\xi}] = 0$, $[\xi\frac{d}{d\xi}, \frac{d}{d\xi}] = -\frac{d}{d\xi}$, $[\frac{d}{d\xi}, \frac{d}{d\xi}] = 0$, is clearly finite dimensional (its dimension equals 2) and is nothing but the Lie superalgebra $\mathbf{W}(1)$ of superderivations of the Grassmann algebra with one variable ξ.

More generally, now let $\Lambda(\xi_1, \ldots, \xi_n)$ be a Grassmann algebra with n generators $\xi_1, \ldots, \xi_n$. Define a *superderivation*

$$\partial : \Lambda(\xi_1, \ldots, \xi_n) \to \Lambda(\xi_1, \ldots, \xi_n)$$

of $\Lambda(\xi_1, \ldots, \xi_n)$ to be a $\mathbb{C}$-linear map that is the sum $\partial_{\bar{0}} + \partial_{\bar{1}}$ of two $\mathbb{C}$-linear maps satisfying the "Leibniz rule with signs"

$$\partial_{\bar{i}}(ab) = \partial_{\bar{i}}(a) \cdot b + (-1)^{\bar{i}\bar{a}} a \cdot \partial_{\bar{i}}(b),$$

where $\bar{i} \in \mathbb{Z}_2$, $a \in \Lambda(\xi_1, \ldots, \xi_n)_{\bar{a}}$, $\bar{a} \in \mathbb{Z}_2$ and $b \in \Lambda(\xi_1, \ldots, \xi_n)$. Then the set $\operatorname{der}\Lambda(\xi_1, \ldots, \xi_n)$ of all superderivations of $\Lambda(\xi_1, \ldots, \xi_n)$ forms a Lie superalgebra denoted by $\mathbf{W}(n)$. (For the reader: please verify that the supercommutator $\partial_{\bar{i}}\partial_{\bar{j}} - (-1)^{\bar{i}\bar{j}}\partial_{\bar{j}}\partial_{\bar{i}}$ is a superderivation for $\bar{i}, \bar{j} \in \mathbb{Z}_2$, and conclude that this defines a Lie superalgebra structure on $\mathbf{W}(n)$.) As an exercise, the reader should also describe the even part $\mathbf{W}(n)_{\bar{0}}$ of $\mathbf{W}(n)$ and show in particular that it is not reductive.

The Lie superalgebra $\mathbf{W}(n)$ is a *Cartan-type* Lie superalgebra and it is simple for $n \geq 3$. There are three more series of Cartan-type Lie superalgebras, $\mathbf{S}$, $\tilde{\mathbf{S}}$, and $\mathbf{H}$, for whose definitions (as well as for the proof of the simplicity of $\mathbf{W}(n)$) we refer to [K2] or [Sch1].

The following is the classification theorem of simple Lie superalgebras proved by V. Kac [K2]. As we mentioned at the beginning of the chapter, this is an important result and its proof is definitely worth studying (see [K2] or [Sch1]).

Theorem 2.4 *The simple Lie superalgebras of the form* $\mathfrak{sl}(V)$, $\mathfrak{osp}(V)$, $\mathfrak{sp}(V)$ *and* $\mathfrak{psq}(V)$, *together with the exceptional Lie algebras and the exceptional Lie superalgebras* $G_{2|1}$, $F_{3|1}$ *and* $D(2, 1; \alpha)$, *as well as the simple Lie superalgebras from the series* $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$ *and* $\mathbf{H}(n)$, *exhaust all simple finite-dimensional Lie superalgebras.*

2.4 Semisimple Lie Superalgebras

We can now state the Kac–Cheng Theorem which characterizes (up to isomorphism) all semisimple finite-dimensional Lie superalgebras. First we need to define the Lie superalgebra $\operatorname{der}\mathfrak{g}$ of superderivations of a Lie superalgebra $\mathfrak{g}$. The definition is analogous to the above definition of superderivations of a Grassmann algebra. A *superderivation* $d : \mathfrak{g} \to \mathfrak{g}$ of a Lie superalgebra $\mathfrak{g}$ is defined to be a linear map that is the sum of two linear maps $d_{\bar{0}}$ and $d_{\bar{1}}$ with the property

$$d_{\bar{i}}([x_1, x_2]) = [d_{\bar{i}}(x_1), x_2] + (-1)^{\bar{i}\bar{x}_1}[x_1, d_{\bar{i}}(x_2)]$$

for $\bar{i} \in \mathbb{Z}_2$ and any homogeneous $x_1 \in \mathfrak{g}_{\bar{x}_1}$ and any $x_2 \in \mathfrak{g}$.

The superderivations of $\mathfrak{g}$ form a Lie superalgebra $\operatorname{der}\mathfrak{g}$. If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, a classical result states that $\operatorname{der}\mathfrak{g} \simeq \mathfrak{g}$. (Prove the theorem yourself, or find and read its proof!) For simple Lie superalgebras this is no longer true, as we will see in the following example.

Example 2.5 Since $\mathfrak{sl}(V)$ is an ideal in $\mathfrak{gl}(V)$, any $x_1 \in \mathfrak{gl}(V)$ defines a superderivation of $\mathfrak{sl}(V)$ by the formula $x_1(x_2) = [x_1, x_2]$, where $x_2 \in \mathfrak{sl}(V)$. This yields a homomorphism of Lie superalgebras $\mathfrak{gl}(V) \to \operatorname{der}\mathfrak{sl}(V)$. The crucial point is that this is a surjection, (see [K2] for a proof), and the kernel is spanned by id_V. Factoring out by the kernel induces an isomorphism

$$\mathfrak{pgl}(V) \cong \operatorname{der}\mathfrak{sl}(V).$$

If $\dim V_{\bar{0}} \neq \dim V_{\bar{1}}$, then the Lie superalgebra $\mathfrak{pgl}(V)$ is isomorphic to $\mathfrak{sl}(V)$ and we have $\operatorname{der}\mathfrak{sl}(V) \cong \mathfrak{sl}(V)$.

Now suppose $n := \dim V_{\bar{0}} = \dim V_{\bar{1}}$. Then $\mathfrak{pgl}(V)$ is not isomorphic to $\mathfrak{sl}(V)$. Moreover, the Lie superalgebra $\mathfrak{pgl}(V)$ is not simple, but it is semisimple when $n \geq 2$. (The reader should show that $\mathfrak{pgl}(V)$ is not isomorphic to a direct sum of two Lie subsuperalgebras.) Finally, since $\operatorname{der}\mathfrak{psl}(V) = \operatorname{der}\mathfrak{sl}(V)$, we conclude that $\operatorname{der}\mathfrak{psl}(V) \cong \mathfrak{pgl}(V)$. ◯

The fundamental paper of Kac [K2] contains a complete list of the Lie superalgebras $\operatorname{der}\mathfrak{s}$ for all simple Lie superalgebras $\mathfrak{s}$ (the cases in which $\operatorname{der}\mathfrak{s} \neq \mathfrak{s}$ are fewer than those in which $\operatorname{der}\mathfrak{s} = \mathfrak{s}$). Furthermore, all such Lie superalgebras $\operatorname{der}\mathfrak{s}$ are semisimple. (Prove!) Therefore, it is no surprise that the Lie superalgebras $\operatorname{der}\mathfrak{s}$ play an important role in the description of all semisimple finite-dimensional Lie superalgebras.

Consider now the following class of Lie superalgebras. Fix simple Lie superalgebras $\mathfrak{s}^1, \ldots, \mathfrak{s}^l$ and non-negative integers $n_1, \ldots, n_l$. Let Λ_{n_i} denote a Grassmann algebra with n_i generators ($\Lambda_0 = \mathbb{C}$). Define the Lie superalgebra

$$\bigoplus_{k=1}^{l} (\operatorname{der}\mathfrak{s}^k \otimes \Lambda_{n_k}) \oplus \left(\bigoplus_{k=1}^{l} \mathbf{W}(n_k) \right) \tag{2.3}$$

by requiring that both $\bigoplus_{k=1}^{l}(\operatorname{der}\mathfrak{s}^k \otimes \Lambda_{n_k})$ and $\bigoplus_{k=1}^{l}\mathbf{W}(n_k)$ be Lie subsuperalgebras and that

$$[d, \sum_{k=1}^{l} s_k \otimes f_k] = (-1)^{\bar{d}\bar{s}_k} d(f_j)$$

for all $d \in \mathbf{W}(n_j)$, $f_k \in \Lambda_{n_k}$ and $\mathbb{Z}_2$-homogeneous elements $s_k \in \operatorname{der}\mathfrak{s}^k$. A straightforward verification shows that (2.3) is a well-defined Lie superalgebra. Furthermore, the Lie superalgebra (2.3) is always semisimple.

The superalgebras (2.3) have natural subsuperalgebras of the form

$$\bigoplus_{k=1}^{l}(\operatorname{der}\mathfrak{s}^k \otimes \Lambda_{n_k}) \dotplus \mathfrak{d} \tag{2.4}$$

where $\mathfrak{d}$ is now an arbitrary Lie subsuperalgebra of $\bigoplus_{k=1}^{l}(\mathbf{W}(n_k))$. Finally, here is the Kac–Cheng Theorem [Ch].

Theorem 2.6 *Every semisimple finite-dimensional Lie superalgebra $\mathfrak{k}$ is isomorphic to a subsuperalgebra $\mathfrak{k}'$ of a Lie superalgebra of the form (2.4), such that*

$$\bigoplus_{k=1}^{l}(\mathfrak{s}^k \otimes \Lambda_{n_k}) \dotplus \mathfrak{d} \subset \mathfrak{k}' \subset \bigoplus_{k=1}^{l}(\operatorname{der}\mathfrak{s}^k \otimes \Lambda_{n_k}) \dotplus \mathfrak{d}$$

for some $\mathfrak{s}^k$ and n_k, with the condition that there are no $\mathfrak{d}$-stable ideals in any of the Grassmann algebras Λ_{n_k}. In addition, all such subsuperalgebras $\mathfrak{k}'$ are semisimple.

2.5 Cartan Subsuperalgebras and Root Systems

As in the case of simple Lie algebras, we would like to list the root systems of (some) simple Lie superalgebras. Before that, we need to briefly discuss Cartan subsuperalgebras of Lie superalgebras.

A *Cartan subsuperalgebra* $\mathfrak{h}$ of any finite-dimensional Lie superalgebra $\mathfrak{g}$ is by definition a self-normalizing nilpotent Lie subsuperalgebra. The following very nice result is due to M. Scheunert.

Theorem 2.7 (Scheunert's Theorem) *The following conditions on a subsuperalgebra $\mathfrak{h} \subset \mathfrak{g}$ are equivalent:*

(1) $\mathfrak{h}$ is a Cartan subsuperalgebra;

(2) $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}_{\overline{0}}} := \{x \in \mathfrak{g} \mid \exists n : (\operatorname{ad} h_1 \circ \cdots \circ \operatorname{ad} h_n)(x) = 0,\ \forall h_1, \ldots, h_n \in \mathfrak{h}_{\overline{0}}\}$;

(3) $\mathfrak{h}_{\overline{0}}$ *is a Cartan subalgebra of* $\mathfrak{g}_{\overline{0}}$ *and* $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}_{\overline{0}}}$.

Theorem 2.7 appeared first in [Sch2] (for the proof, see also [PS4]) and it is important since it shows that the "$\mathfrak{g}$-approach" to Cartan subsuperalgebras (statement *(1)*) and the "$\mathfrak{g}_{\overline{0}}$-approach" (statement *(3)*) yield the same result. Furthermore, it is easy to show that all subsuperalgebras satisfying *(3)* are conjugate under the group of inner automorphisms of $\mathfrak{g}$. More precisely, all Cartan subsuperalgebras are conjugate under the group of so-called elementary automorphisms $\operatorname{Aut}_e \mathfrak{g}_{\overline{0}}$ see [B, Ch.VII, §3, no.2]. The reader can check that there is a natural embedding of $\operatorname{Aut}_e \mathfrak{g}_{\overline{0}}$ into the group of inner automorphisms of $\mathfrak{g}$.

For any finite-dimensional Lie superalgebra $\mathfrak{g}$ and any Cartan subsuperalgebra $\mathfrak{h}$, one has an $\mathfrak{h}_{\overline{0}}$-module decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \neq 0} \mathfrak{g}^\alpha \right) \tag{2.5}$$

with *root spaces*

$$\mathfrak{g}^\alpha := \left\{ x \in \mathfrak{g} \;\middle|\; \begin{array}{c} \exists n : (\operatorname{ad} h_1 - \alpha(h_1)) \circ \cdots \circ (\operatorname{ad} h_n - \alpha(h_n))(x) = 0, \\ \forall h_1, \ldots, h_n \in \mathfrak{h}_{\overline{0}} \end{array} \right\}$$

where $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$ is the linear operator $[h, \cdot\,]$. Statement *(2)* in Scheunert's Theorem implies (prove!) that each $\mathfrak{g}^\alpha$ is not only an $\mathfrak{h}_{\overline{0}}$-module, but is in fact an $\mathfrak{h}$-module. Therefore it is natural to call decomposition (2.5) the *generalized* $\mathfrak{h}$*-root decomposition* of $\mathfrak{g}$, or simply the *generalized root decomposition* of $\mathfrak{g}$, and to call the set

$$\Delta := \{\alpha \in \mathfrak{h}_{\overline{0}}^* \mid \alpha \neq 0 \text{ and } \mathfrak{g}^\alpha \neq 0\}$$

the $\mathfrak{h}$*-root system* of $\mathfrak{g}$, or simply the *root system* of $\mathfrak{g}$. Thus we can write (2.5) as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha \right). \tag{2.6}$$

We are now ready to write down the root systems of all simple Lie superalgebras of classical and exceptional types, as well as that of $\mathbf{W}(n)$. The Lie superalgebras of type $A_{m-1|n-1}$ are simple for $(m, n) \neq (0, 0), (0, 1), (1, 0), (1, 1)$; the Lie superalgebras of type $B_{m|n}$ are simple for $(m, n) \neq (1, 0), (1, 1), (3, 0), (3, 1)$; the Lie superalgebras of type $C_{1|n}$ are simple for $n \geq 1$; and the Lie superalgebras of type $D_{m|n}$ are simple for $(m, n) \neq (0, 2)$. For $\mathfrak{g}$ of type $A_{m-1|n-1}$, $B_{m|n}$, $C_{1|n}$, $D_{m|n}$ and for the exceptional Lie superalgebras, we let $(\cdot, \cdot)$ denote a nondegenerate even supersymmetric invariant bilinear form on $\mathfrak{h}^*$.

Fix the vector space $\mathbb{C}^{m+n}$ with the trivial $\mathbb{Z}_2$-grading (i.e., $(\mathbb{C}^{m+n})_{\overline{1}} = 0$) and denote its standard basis by $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n$, where for $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|n)$ and the exceptional Lie superalgebras, $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$ and $(\varepsilon_i, \delta_j) = 0$. The standard coordinates are denoted respectively by $a_1, \ldots, a_m$, $b_1, \ldots, b_n$. In all cases $\mathfrak{h}_{\overline{0}}^* = \operatorname{span}\Delta$, and $\operatorname{span}\Delta = \mathbb{C}^{m+n}$ except for $A_{m-1|n-1}$, $G_{2|1}$, $\mathfrak{sp}(V)$ and $\mathfrak{psq}(V)$. Furthermore, in all cases except for $A_{1,1}$, $\mathfrak{psq}(V)$ and $\mathbf{W}(n)$, each root space $\mathfrak{g}^\alpha$ has dimension 1. In addition, for $A_{1,1}$ and $\mathbf{W}(n)$, it is also true that $(\mathfrak{g}^\alpha)_{\overline{0}} = 0$ or $(\mathfrak{g}^\alpha)_{\overline{1}} = 0$ for any root $\alpha \in \Delta$. So for $\mathfrak{g} \neq \mathfrak{psq}(V)$, we can define the *even roots* of $\mathfrak{g}$ to be

$$\Delta_{\overline{0}} := \{\alpha \in \Delta \mid (\mathfrak{g}^\alpha)_{\overline{1}} = 0\}$$

and the *odd roots* of $\mathfrak{g}$ as

$$\Delta_{\overline{1}} := \{\alpha \in \Delta \mid (\mathfrak{g}^\alpha)_{\overline{0}} = 0\}.$$

Here is a list of the root systems:

$A_{m-1|n-1}$: $m \neq n$, $\Delta = \Delta_{\overline{0}} \sqcup \Delta_{\overline{1}} \subset \mathbb{C}^{m+n}$,

$$\Delta_{\overline{0}} = \left\{\varepsilon_i - \varepsilon_j,\ \delta_k - \delta_\ell \mid 1 \leq i \neq j \leq m,\ 1 \leq k \neq \ell \leq n\right\},$$

$$\Delta_{\overline{1}} = \{\pm(\varepsilon_i - \delta_k) \mid 1 \leq i \leq m,\ 1 \leq k \leq n\}$$

$$\text{span}\,\Delta \text{ is the hyperplane with equation } \sum_{i=1}^{m} a_i + \sum_{k=1}^{n} b_k = 0;$$

$A_{n-1|n-1}$: $m = n$, $\Delta = \Delta_{\overline{0}} \sqcup \Delta_{\overline{1}} \subset \mathbb{C}^{m+n}/\operatorname{span}\{\mathbf{a}\}$,

where $\mathbf{a}$ is the vector with coordinates $a_i = 1, b_i = -1$ for $1 \leq i \leq m$. The sets $\Delta_{\overline{0}}$ and $\Delta_{\overline{1}}$ are the projections of the corresponding sets defined above (where now $m = n$) into $\mathbb{C}^{m+n}/\operatorname{span}\{\mathbf{a}\}$, and span Δ is the image of the hyperplane with equation $\displaystyle\sum_{i=1}^{m}(a_i + b_i) = 0$;

$B_{m|n}$: $\Delta = \Delta_{\overline{0}} \sqcup \Delta_{\overline{1}} \subset \mathbb{C}^{m+n}$,

$$\Delta_{\overline{0}} = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i,\ \pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k \;\middle|\; \begin{matrix} 1 \leq i < j \leq m, \\ 1 \leq k < \ell \leq n \end{matrix}\right\},$$

$$\Delta_{\overline{1}} = \{\pm\delta_k \pm \varepsilon_i,\ \pm\delta_k \mid 1 \leq i \leq m,\ 1 \leq k \leq n\};$$

$C_{1|n}$: $m = 1,\ \Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}} \subset \mathbb{C}^{1+n}$,

$\Delta_{\bar{0}} = \{\pm\delta_k \pm \delta_\ell,\ \pm 2\delta_k \mid 1 \le k < \ell \le n\}$,

$\Delta_{\bar{1}} = \{\pm\varepsilon_1 \pm \delta_k \mid 1 \le k \le n\}$;

$D_{m|n}$: $\Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}} \subset \mathbb{C}^{m+n}$,

$\Delta_{\bar{0}} = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\delta_k \pm \delta_\ell, \pm 2\delta_k \mid 1 \le i < j \le m,\ 1 \le k < \ell \le n\right\}$,

$\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_k \mid 1 \le i \le m,\ 1 \le k \le n\}$;

$G_{2|1}$: $m = 3,\ n = 1,\ \Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}} \subset \mathbb{C}^{3+1}$,

$\Delta_{\bar{0}} = \left\{\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i,\ \pm 2\delta_1 \mid 1 \le i \ne j \le 3\right\}$,

$\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta_1,\ \pm\delta_1 \mid 1 \le i \le 3\}$,

span Δ is the hyperplane with equation $\displaystyle\sum_{i=1}^{3} a_i = 0$;

$F_{3|1}$: $m = 3,\ n = 1,\ \Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}} \subset \mathbb{C}^{3+1}$,

$\Delta_{\bar{0}} = \left\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i, \pm\delta_1 \mid 1 \le i < j \le 3\right\}$,

$\Delta_{\bar{1}} = \left\{\dfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta_1)\right\}$;

$D(2, 1; \alpha)$: $m = 3,\ n = 0,\ \Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}}$,

$\Delta_{\bar{0}} = \{\pm 2\varepsilon_i \mid 1 \le i \le 3\}$,

$\Delta_{\bar{1}} = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}$,

(Δ does not depend on the complex parameter α);

$\mathfrak{sp}(V)$: $\dim V_{\bar{0}} = \dim V_{\bar{1}} = m \ge 3,\ n = 0$,

$\Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}} \subset \mathbb{C}^m / \operatorname{span}\{(1, 1, \ldots, 1)\}$,

$\Delta_{\bar{0}} = \left\{\tilde{\varepsilon}_i - \tilde{\varepsilon}_j \mid 1 \le i \ne j \le m\right\}$,

$\Delta_{\bar{1}} = \left\{\pm(\tilde{\varepsilon}_i + \tilde{\varepsilon}_j),\ 2\tilde{\varepsilon}_i \mid 1 \le i \ne j \le m\right\}$,

where $\tilde{\varepsilon}_i$ is the image of ε in $\mathbb{C}^m / \operatorname{span}\{(1, \ldots, 1)\}$;

$\mathfrak{psq}(V)$: $\dim V_{\bar{0}} = \dim V_{\bar{1}} = m \ge 3,\ n = 0$,

$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le m\}$,

span Δ is the hyperplane with equation $\displaystyle\sum_{i=1}^{m} a_i = 0$,

$\dim(\mathfrak{g}^\alpha)_{\bar{0}} = \dim(\mathfrak{g}^\alpha)_{\bar{1}} = 1\ \forall \alpha \in \Delta$;

$$\mathbf{W}(n):\ m=0,\ \Delta=\Delta_{\bar{0}}\sqcup\Delta_{\bar{1}}\subset\mathbb{C}^n,$$

$$\Delta_{\bar{0}}=\left\{\begin{array}{l|l}\varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_k}-\varepsilon_j & \\ \varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_r} & \end{array}\right.$$

$$\left.\begin{array}{l}0\le k<n,\,k\text{ odd},\,i_1<i_2<\cdots<i_k,\\ 1\le j\le n,\,j\neq i_t\text{ for }t=1,\ldots,k,\\ 1\le r<n,\,r\text{ even},\,i_1<i_2<\cdots<i_r\end{array}\right\},$$

$$\Delta_{\bar{1}}=\left\{\begin{array}{l|l}\varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_k}-\varepsilon_j & \\ \varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_r} & \end{array}\right.$$

$$\left.\begin{array}{l}0\le k<n,\,k\text{ even},\,i_1<i_2<\cdots<i_k,\\ 1\le j\le n,\,j\neq i_t\text{ for }t=1,\ldots,k,\\ 1\le r<n,\,r\text{ odd},\,i_1<i_2<\cdots<i_r\end{array}\right\}.$$

For a description of the root systems of the remaining simple finite-dimensional Lie superalgebras $S(n)$, $\tilde{S}(n)$, $H(n)$, see [P3].

2.6 Borel Subsuperalgebras and Sets of Simple Roots

We will complete this chapter with a short discussion of Borel subsuperalgebras of Lie superalgebras. The general question of finding a natural generalization of the notion of a Borel subalgebra of a finite-dimensional Lie algebra to an arbitrary finite-dimensional Lie superalgebra seems to be open. However for certain Lie superalgebras whose structure is sufficiently well understood, there is an established version of such a definition. It has been used for simple Lie superalgebras since the inception of the theory in the 1970s, and we describe it now.

Let $\mathfrak{g}$ be any finite-dimensional Lie superalgebra. Fix a Cartan subsuperalgebra $\mathfrak{h}=\mathfrak{h}_{\bar{0}}\oplus\mathfrak{h}_{\bar{1}}$ of $\mathfrak{g}$, and consider the generalized root decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus\left(\bigoplus_{\alpha\in\Delta}\mathfrak{g}^\alpha\right).$$

Since Δ is a finite set, a generic hyperplane (i.e., a vector subspace of codimension 1 in general position) in the real vector space $\mathrm{span}_{\mathbb{R}}\,\Delta$ is regular, i.e., it does not intersect Δ. Any regular hyperplane defines a decomposition of Δ into a positive and negative part $\Delta=\Delta^+\sqcup\Delta^-$, as the hyperplane cuts $\mathrm{span}_{\mathbb{R}}\,\Delta$ into two connected

components. (Prove!) The designations + and − can be chosen arbitrarily. The ($\mathbb{Z}_2$-graded vector space compatible) decomposition

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in\Delta^-} \mathfrak{g}^\alpha\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha\right)$$

is by definition a *triangular decomposition* of $\mathfrak{g}$.

As a working definition, one can think of the solvable Lie subsuperalgebras of the form $\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha)$ as *Borel subsuperalgebras* of $\mathfrak{g}$, and in this case $\Delta^+ = \Delta_\mathfrak{b}$. The obvious problem with this definition is that it does not coincide with the definition of a Borel subalgebra for a general finite-dimensional Lie algebra that was given in Sect. 1.3. However for a reductive Lie algebra our working definition is clearly equivalent to the standard definition of Borel subalgebra. Moreover, for a classical or exceptional Lie superalgebra $\mathfrak{g} \not\simeq \mathfrak{sp}(n)$, the above working definition is equivalent to the characterization from Sect. 1.3 of a Borel subalgebra via their roots (see conditions (B1) and (B2). This equivalence will also hold for $\mathfrak{g} = \mathfrak{sp}(n)$ after we replace (B2) by

(B2′) for any $\alpha \in \Delta$ such that $-\alpha \in \Delta$, exactly one of the following possibilities holds: $\alpha \in \Delta_\mathfrak{b}$ or $-\alpha \in \Delta_\mathfrak{b}$.

Clearly, the so-defined Borel subsuperalgebras are solvable, but they are not necessarily maximal solvable. Here is the simplest example in which this appears.

Example 2.8 Let $\mathfrak{g}$ be of type $A_{1,0}$. Then $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ is two-dimensional and $\Delta = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \delta_1), \pm(\varepsilon_2 - \delta_1)\}$. The Lie subsuperalgebra $\mathfrak{p} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta\setminus\{\varepsilon_2-\varepsilon_1,\delta_1-\varepsilon_1\}} \mathfrak{g}^\alpha)$ is solvable and it properly contains the Borel subsuperalgebra $\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha=\varepsilon_1-\varepsilon_2,\varepsilon_1-\delta_1,\varepsilon_2-\delta_1} \mathfrak{g}^\alpha)$. Indeed, here is a sequence of subsuperalgebras in $\mathfrak{p}$ such that each subsuperalgebra is an ideal in the next one and all quotients are abelian:

$$\mathfrak{p}' := \mathfrak{g}^{\varepsilon_1-\varepsilon_2} \oplus \mathfrak{g}^{\varepsilon_1-\delta_1} \subset \mathfrak{p}'' := \mathfrak{h}' \oplus \mathfrak{p}' \subset \mathfrak{p}''' := \mathfrak{p}'' \oplus \mathfrak{g}^{\varepsilon_2-\delta_1} \oplus \mathfrak{g}^{\delta_1-\varepsilon_2} \subset \mathfrak{p}'''' := \mathfrak{p},$$

where $\mathfrak{h}' = \{h \in \mathfrak{h} \mid \varepsilon_1(h) = 0\}$. ◯

Simple roots of Borel subsuperalgebras play as important a role in the theory of Lie superalgebras as they do in the theory of Lie algebras, but their nature is quite different. First of all with the above definition of a Borel subsuperalgebra $\mathfrak{b}$, a set $\sum_\mathfrak{b}$ of *simple roots* for $\mathfrak{b}$ is defined exactly as in Chap. 1, i.e., $\sum_\mathfrak{b}$ is the set of roots $\alpha \in \Delta^+$ that cannot be decomposed as the sum of two roots in Δ^+. A new feature is that one and the same Lie superalgebra $\mathfrak{g}$ can have Borel subsuperalgebras that are not isomorphic as Lie superalgebras and their sets of simple roots can differ significantly. This phenomenon is evident already when $\mathfrak{g}$ is of type $A_{m-1|n-1}$, as we will now see.

Suppose that $\mathfrak{g}$ is of type $A_{m-1|n-1}$, and fix a Cartan subsuperalgebra $\mathfrak{h}$ in $\mathfrak{g}$ (equivalently a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\overline{0}}$ in $\mathfrak{g}_{\overline{0}} = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}$), and let $\mathfrak{b}'$ be the Borel subalgebra of $\mathfrak{g}_{\overline{0}}$ with roots

$$\{\varepsilon_i - \varepsilon_j\}_{1 \le i < j \le m} \cup \{\delta_k - \delta_\ell\}_{1 \le k < \ell \le n}.$$

Consider the set of vectors $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n$ as in the description of the roots of $A_{m,n}$, and let $\kappa_1, \ldots, \kappa_{m+n}$ be any strict linear ordering of this set with the property that the orderings that it induces on the subset of "epsilons" and "deltas" are the initial ones, i.e., $\varepsilon_1, \ldots, \varepsilon_m$ and $\delta_1, \ldots, \delta_n$. Then the simple roots of an arbitrary Borel subsuperalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{b}_{\overline{0}} = \mathfrak{b}'$ are precisely the elements

$$\kappa_1 - \kappa_2, \kappa_2 - \kappa_3, \ldots, \kappa_{m+n-1} - \kappa_{m+n}$$

for any such ordering. An immediate consequence of this is that $A_{m-1|n-1}$ will be described by many different Dynkin diagrams! In this book, we will not discuss the notion of a Dynkin diagram for a Lie superalgebra (it can be found in [K2] and in the physics literature); instead we will restrict ourselves to several comments on the other simple Lie superalgebras.

The description of the simple roots for Borel subsuperalgebras of Lie superalgebras of types $B_{m|n}$, $C_{1|m}$, and $D_{m|n}$ is similar, but slightly more complicated. On the other hand, the description of simple roots for $\mathfrak{psq}(V)$ is straightforward: here a Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}_{\overline{0}}$ determines a unique Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$ with $\mathfrak{b}_{\overline{0}} = \mathfrak{b}'$, and the simple roots of $\mathfrak{b}$ are just the simple roots of $\mathfrak{b}'$. For the Lie superalgebra $\mathfrak{sp}(V)$ a new phenomenon occurs, namely that not all Borel subsuperalgebras have the same dimension. Indeed, consider the following example.

Example 2.9 Let $\mathfrak{g} = \mathfrak{sp}(V)$, with $\dim V_{\overline{0}} = 3$. The Borel subsuperalgebra of $\mathfrak{sp}(V)$ with simple roots $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, 2\varepsilon_3\}$ has dimension 11, while the opposite Borel subsuperalgebra with simple roots $\{-\varepsilon_3 - \varepsilon_2, \varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1\}$ has dimension 8. ◯

Example 2.10 The case $\mathfrak{g} = \mathbf{W}(n)$ is most interesting. Here the root spaces can have dimension greater than 1. For instance the root space $\mathfrak{g}^{\varepsilon_1}$ equals

$$\operatorname{span}\left\{\xi_1\xi_2\frac{\partial}{\partial\xi_2}, \ldots, \xi_1\xi_n\frac{\partial}{\partial\xi_n}\right\}$$

and consequently has dimension $n-1$ (more precisely, $\dim(\mathfrak{g}^{\varepsilon_1})_{\overline{0}} = 0$, $\dim(\mathfrak{g}^{\varepsilon_1})_{\overline{1}} = n - 1$). A Borel subsuperalgebra of maximal dimension in $\mathbf{W}(n)$ is the Borel subsuperalgebra with simple roots

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\},$$

and its opposite Borel subsuperalgebra, with simple roots

$$\{-\varepsilon_n, \varepsilon_n - \varepsilon_{n-1}, \varepsilon_{n-1} - \varepsilon_{n-2} \ldots, \varepsilon_2 - \varepsilon_1\}$$

has minimal dimension. We note also that, contrary to the case of simple Lie superalgebras of classical or exceptional type, for $\mathfrak{g} = \mathbf{W}(n)$ it is no longer true that if $\mathfrak{b}$ is a Borel subsuperalgebra of $\mathfrak{g}$, then $\mathfrak{b}_{\bar{0}}$ is a Borel subalgebra of $\mathfrak{g}_{\bar{0}}$. ◯

Whenever $\mathfrak{g}_{\bar{0}}$ is reductive, we let W be the Weyl group of $\mathfrak{g}_{\bar{0}}$. The above examples show that one cannot expect W to act transitively on the sets of simple roots of a Lie superalgebra $\mathfrak{g}$, and generally this fails for $\mathfrak{g} \neq \mathfrak{psq}(n), \mathfrak{osp}(1|2n)$. However, by introducing "odd" reflections, one can "move" between the various Borel subsuperalgebras of the classical and exceptional Lie superalgebras (that contain $\mathfrak{h}$), as we will explain here.

We say that two Borel subsuperalgebras $\mathfrak{b}$, $\mathfrak{b}'$ of $\mathfrak{g}$ are *adjacent* if

$$\left(\Delta^+(\mathfrak{b}) \setminus \Delta^+(\mathfrak{b}')\right) \subset \{\alpha, 2\alpha\} \text{ and } \left(\Delta^+(\mathfrak{b}') \setminus \Delta^+(\mathfrak{b})\right) \subset \{-\alpha, -2\alpha\}$$

for some $\alpha \in \Delta$. Note that α is necessarily simple for $\mathfrak{b}$, and if $-\alpha \in \Delta$, then $-\alpha$ is simple for $\mathfrak{b}'$. In the case that α is even, $w_\alpha(\Delta^+(\mathfrak{b})) = \Delta^+(\mathfrak{b}')$ where $w_\alpha \in W$ is the reflection at α. If $2\alpha \in \Delta$, then α is odd, 2α is even and $w_{2\alpha}(\Delta^+(\mathfrak{b})) = \Delta^+(\mathfrak{b}')$. If α is odd and $2\alpha \notin \Delta$, then we define the *odd reflection* r_α by setting $r_\alpha(\Delta^+(\mathfrak{b})) := \Delta^+(\mathfrak{b}')$ and $r_\alpha(\Delta^+(\mathfrak{b}')) := \Delta^+(\mathfrak{b})$, and we say that the Borel subsuperalgebras $\Delta^+(\mathfrak{b})$ and $\Delta^+(\mathfrak{b}')$ are *connected by an odd reflection*. We call an odd reflection r_α *degenerate* if $-\alpha \notin \Delta$.

The following is a well-known fact.

Lemma 2.11 *Let $\mathfrak{g}$ be a classical or exceptional Lie superalgebra, and fix a Borel subalgebra $\mathfrak{b}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$. Any two Borel subsuperalgebras $\mathfrak{b}$, $\mathfrak{b}'$ of $\mathfrak{g}$ containing $\mathfrak{b}_{\bar{0}}$ can be connected by a chain of odd reflections. In the case of $\mathbf{p}(n)$ these odd reflections may be degenerate.*

Proof The proof is by induction on the number d of roots in

$$(\Delta^+(\mathfrak{b}) \setminus \Delta^+(\mathfrak{b}')) \cup (\Delta^+(\mathfrak{b}') \setminus \Delta^+(\mathfrak{b})).$$

If this set is empty, then $\Delta^+(\mathfrak{b}) = \Delta^+(\mathfrak{b}')$, and we are done. Suppose it is not.

Without loss of generality, we may assume $\Delta^+(\mathfrak{b}) \setminus \Delta^+(\mathfrak{b}') \neq \emptyset$. Then there exists a simple root β of $\Delta^+(\mathfrak{b})$ such that $\beta \notin \Delta^+(\mathfrak{b}')$, since otherwise we would have $\Delta^+(\mathfrak{b}) \subset \Delta^+(\mathfrak{b}')$. It follows that β is odd and that 2β is not a root of $\mathfrak{g}$ as the even roots of $\Delta^+(\mathfrak{b})$ and $\Delta^+(\mathfrak{b}')$ coincide by the assumption $\mathfrak{b}_{\bar{0}} = \mathfrak{b}'_{\bar{0}}$.

For $\mathfrak{g} \not\simeq \mathfrak{sp}(n)$, we define the odd reflection r_β by

$$r_\beta(\Delta^+(\mathfrak{b})) := \begin{cases} (\Delta^+(\mathfrak{b}) \setminus \{\beta\}) \sqcup \{-\beta\} & \text{if } -\beta \in \Delta, \\ \Delta^+(\mathfrak{b}) \setminus \{\beta\} & \text{if } -\beta \notin \Delta \end{cases}$$

and claim that $\Delta^+ := r_\beta(\Delta^+(\mathfrak{b}))$ is the set of roots of a Borel subsuperalgebra of $\mathfrak{g}$. Clearly, Δ^+ satisfies condition (B2′) as $\Delta^+(\mathfrak{b})$ satisfies (B2′). Since β is simple for $\Delta^+(\mathfrak{b})$, to prove condition (B1) for Δ^+, it suffices to show that if $\alpha \in (\Delta^+(\mathfrak{b}) \setminus \{\beta\})$ and $\alpha - \beta \in \Delta$, then $\alpha - \beta \in \Delta^+$. As $\mathfrak{g} \not\simeq \mathfrak{sp}(n)$, every root is either a non-negative or a non-positive linear combination of the simple roots of $\Delta^+(\mathfrak{b})$. So the claim follows immediately from expressing α as a (non-negative) linear combination of the simple roots of $\Delta^+(\mathfrak{b})$. Indeed, since β is simple for $\Delta^+(\mathfrak{b})$, if $\alpha - \beta \in \Delta$, then $\alpha - \beta$ is (still) a non-negative linear combination of the simple roots of $\Delta^+(\mathfrak{b})$, and hence $\alpha - \beta \in \Delta^+(\mathfrak{b}) \setminus \{\beta\}$.

Now let $\mathfrak{g} = \mathfrak{sp}(n)$. The root system of $\mathfrak{sp}(n)$ is a subset of the root system of the Lie algebra $\mathfrak{sp}(2n)$, where the parity is disregarded. This induces a 1-1 correspondence between the Borel subsuperalgebras of $\mathfrak{sp}(n)$ (containing a fixed Cartan subsuperalgebra) and the Borel subalgebras of $\mathfrak{sp}(2n)$ (containing a fixed Cartan subalgebra). An odd reflection for $\mathfrak{sp}(n)$ as defined above corresponds to a usual Weyl group reflection w_β for $\mathfrak{sp}(2n)$, and is degenerate when $-\beta \notin \Delta$. Hence, if $\tilde{\mathfrak{b}}$ is the image of $\mathfrak{b}$ under this correspondence, then $w_\beta(\Delta^+(\tilde{\mathfrak{b}}))$ is the set of roots of a Borel subalgebra of $\mathfrak{sp}(2n)$, and so $r_\beta(\Delta^+(\mathfrak{b})) := w_\beta(\Delta^+(\tilde{\mathfrak{b}})) \cap \Delta$ the set of roots of a Borel subsuperalgebra of $\mathfrak{sp}(n)$.

Clearly, in all cases of $\mathfrak{g}$, the cardinality of the set $(\Delta^+ \setminus \Delta^+(\mathfrak{b}')) \cup (\Delta^+(\mathfrak{b}') \setminus \Delta^+)$ is strictly less than d. Since Δ is finite and odd reflections are invertible, induction completes the proof. □

Definition 2.12 For each Lie superalgebra $\mathfrak{g}$, we call the following set of simple roots and corresponding Borel subsuperalgebra *standard* [K2, Table VI]:

$$
\begin{aligned}
A_{m-1|n-1} \quad & \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\},\\
B_{m|n} \quad & \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\},\\
C_{1|n} \quad & \{\varepsilon_1 - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\},\\
D_{m|n} \quad & \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\},\\
G_{2|1} \quad & \{\delta_1 + \varepsilon_1, \varepsilon_2, \varepsilon_3 - \varepsilon_2\},\\
F_{3|1} \quad & \left\{\frac{1}{2}(\delta_1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3), -\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\right\},\\
D(2,1;\alpha) \quad & \{\varepsilon_1 + \varepsilon_2 + \varepsilon_3, -2\varepsilon_2, -2\varepsilon_3\},\\
\mathfrak{sp}(n) \quad & \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\},\\
\mathfrak{psq}(n) \quad & \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\}.
\end{aligned}
$$

A *parabolic subsuperalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is by definition a subsuperalgebra of $\mathfrak{g}$ that contains a Borel subsuperalgebra. Note that in contrast with the case of a Lie algebra, a parabolic subsuperalgebra may be solvable even if it properly contains a Borel subsuperalgebra: we saw this in Example 2.8.

We conclude this chapter with the remark that the theory of simple Lie superalgebras has some parallels to the theory of simple finite-dimensional Lie algebras in characteristic p. In particular, the Cartan-type simple Lie superalgebras are direct analogs of the Cartan-type simple Lie algebras in characteristic p, which are well known. Nevertheless, only a rough parallel between the representation theories of Lie superalgebras and Lie algebras in characteristic p seems to exist. Both these representation theories are currently very active subjects (which remain largely beyond the scope of this book), and interested readers may try to form their own opinions on the strength of this analogy.

Chapter 3
Root-Reductive Lie Algebras

In this chapter we introduce the main object of our study, the infinite-dimensional root-reductive Lie algebras.

3.1 Definitions

A Lie algebra $\mathfrak{g}$ is *locally finite* if every finite set of elements $x_1, \ldots, x_k \in \mathfrak{g}$ is contained in a finite-dimensional Lie subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ is countable dimensional, then being locally finite is equivalent to being isomorphic to a union

$$\bigcup_{n\in\mathbb{Z}_{>0}} \mathfrak{g}_n, \tag{3.1}$$

of nested finite-dimensional Lie algebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n \subset \mathfrak{g}_{n+1} \subset \cdots .$$

We call (3.1) an *exhaustion* for $\mathfrak{g}$. If $\mathfrak{g}_n = \mathfrak{g}_{n+1} = \mathfrak{g}_{n+2} = \cdots$ for some n, then $\mathfrak{g}$ is finite dimensional, whereas if the $\mathfrak{g}_n$'s do not stabilize, then $\mathfrak{g}$ is infinite dimensional.

The first thing that comes to mind when one wants to construct an example of an infinite-dimensional locally finite Lie algebra is to let $\mathfrak{g}$ equal the union of nested subalgebras $\mathfrak{g}_n \cong \mathfrak{sl}(n)$ for $n \in \mathbb{Z}_{>0}$; in other words, $\mathfrak{g}$ is the direct limit $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ for $\mathfrak{g}_n \cong \mathfrak{sl}(n)$. This case is indeed quite straightforward as any Lie algebra embedding of the form $\mathfrak{sl}(n) \subset \mathfrak{sl}(n+1)$ for $n > 2$ is conjugate to a lower right- or an upper left-hand corner inclusion of matrices,

$$A \mapsto \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \quad \text{or} \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_3

and up to isomorphism, the Lie algebra obtained as this union does not depend on the particular choice of inclusions. (To the reader: as an exercise try to prove the latter statement on your own; we will give a proof later in this chapter.) The resulting Lie algebra is by definition the Lie algebra $\mathfrak{sl}(\infty)$.

However, if one just relaxes a bit the condition on $\mathfrak{g}_n$ and requires $\mathfrak{g}_n$ to be isomorphic to $\mathfrak{sl}(i_n)$ for some increasing sequence of positive integers i_n with $\lim_{n\to\infty} i_n = \infty$, then the situation becomes much more complicated, and there is no classification of the resulting Lie algebras. The reason is that there are many non-equivalent (i.e., non-conjugate via an automorphism) embeddings of $\mathfrak{sl}(i_n)$ into $\mathfrak{sl}(i_{n+1})$ when $i_{n+1} - i_n$ is large. By varying the embeddings of $\mathfrak{sl}(i_n)$ into $\mathfrak{sl}(i_{n+1})$, one can obtain a variety of non-isomorphic locally finite infinite-dimensional Lie algebras. We will discuss some of them in Chap. 4.

In order to single out the simplest class of locally finite Lie algebras similar to $\mathfrak{sl}(\infty)$, we give the following definition. An injection of finite-dimensional Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ is called a *root injection* if for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the subspace $\mathfrak{h}' := \mathfrak{k} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$ and each $\mathfrak{h}'$-root space $\mathfrak{k}^\alpha$ is an $\mathfrak{h}$-root space of $\mathfrak{g}$, (i.e., $\mathfrak{k}^\alpha = \mathfrak{g}^{\tilde{\alpha}}$ for some $\tilde{\alpha} \in \mathfrak{h}^*$). Somewhat informally we can say that each root space of $\mathfrak{k}$ is being mapped into a (single) root space of $\mathfrak{g}$ under the injection.

A locally finite Lie algebra $\mathfrak{g}$ is called *root-reductive* if it is isomorphic to a union $\bigcup_{n\in\mathbb{Z}_{>0}} \mathfrak{g}_n$ of finite-dimensional reductive subalgebras with respect to root injections $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ for a fixed choice of nested Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{g}_n$ satisfying $\mathfrak{h}_{n-1} = \mathfrak{h}_n \cap \mathfrak{g}_{n-1}$ for $n \in \mathbb{Z}_{>0}$.

The usual finite-dimensional reductive Lie algebras $\mathfrak{g}$ are root-reductive for a stable sequence $\mathfrak{g}_n = \mathfrak{g}$ for all n. The Lie algebra $\mathfrak{sl}(\infty)$ is also root-reductive with respect to the lower right-hand or upper left-hand corner inclusions $\mathfrak{sl}(n) \subset \mathfrak{sl}(n+1)$. These same inclusions obviously extend to $\mathfrak{gl}(n)$, i.e., $\mathfrak{gl}(n) \subset \mathfrak{gl}(n+1)$, and the resulting infinite-dimensional Lie algebra does not depend, up to isomorphism, on the choice of lower right-hand corner or upper left-hand corner inclusions, to name just two possibilities. This Lie algebra is denoted by $\mathfrak{gl}(\infty)$.

Since every finite-dimensional reductive Lie algebra $\mathfrak{g}_n$ decomposes canonically as a direct sum of simple components plus a center, it is natural to ask whether the same result holds for root-reductive Lie algebras. The answer is "no"; however, a weaker version of the result does in fact extend to arbitrary root-reductive Lie algebras. Let's look first at $\mathfrak{gl}(\infty)$.

Somewhat counter-intuitively (but only at first glance), the Lie algebra $\mathfrak{gl}(\infty)$ is not isomorphic to a direct sum $\mathfrak{sl}(\infty) \oplus \mathbb{C}$, as the center $Z_{\mathfrak{gl}(\infty)}$ of $\mathfrak{gl}(\infty)$ is trivial, i.e., $Z_{\mathfrak{gl}(\infty)} = 0$. This follows from the obvious remark that $Z_{\mathfrak{gl}(n)}$ does not map into $Z_{\mathfrak{gl}(n+1)}$ under the upper left- or lower right-hand corner inclusion. Consequently, no nonzero element of $\mathfrak{gl}(\infty)$ commutes with $\mathfrak{gl}(\infty)$. On the other hand, there is a canonical injection $\mathfrak{sl}(\infty) \to \mathfrak{gl}(\infty)$, as the derived Lie algebra $[\mathfrak{gl}(\infty), \mathfrak{gl}(\infty)]$ is clearly isomorphic to $\mathfrak{sl}(\infty)$. Therefore the isomorphism

$$\mathfrak{gl}(n) \simeq \mathfrak{sl}(n) \oplus \mathbb{C},$$

when taken to the limit $n \to \infty$, becomes a non-split exact sequence of Lie algebras

$$0 \to \mathfrak{sl}(\infty) \to \mathfrak{gl}(\infty) \to \mathbb{C} \to 0. \tag{3.2}$$

This exact sequence (3.2) can be generalized to a description of arbitrary root-reductive Lie algebras. We will state this general description in Sect. 3.3.

An immediate consequence of the definition of a root-reductive Lie algebra is that it has a root decomposition of the form

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha}), \tag{3.3}$$

where $\mathfrak{h} := \bigcup_n \mathfrak{h}_n$ ($\mathfrak{h}_n$ being the fixed Cartan subalgebra of $\mathfrak{g}_n$), and each $\mathfrak{g}^{\alpha}$ is the image in $\mathfrak{g}$ of the direct limit of a sequence of root spaces $\mathfrak{g}_n^{\alpha_n}$, each root space being mapped into the next via the root injections defining $\mathfrak{g}$. Clearly, $\dim \mathfrak{g}^{\alpha} = 1$ for every $\alpha \in \Delta$; furthermore, α stands here for the inverse limit of the respective linear functions $\alpha_n : \mathfrak{h}_n \to \mathbb{C}$ and is a well-defined linear function on $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$. By definition, Δ is the set of all linear functions on $\mathfrak{h}$ obtained in this way (which are automatically nonzero functions) and Δ is called the set of *roots* of $\mathfrak{g}$. Furthermore, clearly if $\alpha \in \Delta$, then $-\alpha \in \Delta$, and $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$ generate an $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$.

An important structural remark is that any root-reductive Lie algebra that is generated by its root spaces is determined up to isomorphism by the pair $\{\mathfrak{h} = \text{span}\,\Delta, \Delta\}$. (The reader should give a precise proof using the corresponding result for finite-dimensional semisimple Lie algebras.) A substantial difference from the finite-dimensional case is that the converse statement does not hold, i.e., the isomorphism class of $\mathfrak{g}$ does not determine a unique pair $\{\text{span}\,\Delta, \Delta\}$ (even up to an isomorphism of pairs). This will become apparent below in the description of B_∞ and D_∞.

3.2 Locally Simple Root-Reductive Lie Algebras

Now consider the special case of a *locally simple* root-reductive Lie algebra $\mathfrak{s} = \bigcup_n \mathfrak{s}_n$, in other words, a Lie algebra $\mathfrak{s}$ that is the union of simple finite-dimensional Lie algebras under root injections. Assume that $\mathfrak{s}$ is infinite dimensional. Then we can choose a subsequence $\mathfrak{s}_{n_k}$ of $\mathfrak{s}_n$ such that the injections $\mathfrak{s}_{n_k} \subset \mathfrak{s}_{n_{k+1}}$ are proper and all $\mathfrak{s}_{n_k}$ are simultaneously of the same type, A, B, C or D. Indeed, one of the non-exceptional types A, B, C and D has to appear infinitely often among the $\mathfrak{s}_n$. Moreover $\mathfrak{s} = \bigcup_k \mathfrak{s}_{n_k}$. Therefore to study the isomorphism classes of arbitrarily locally simple root-reductive Lie algebras, it suffices to analyze sequences of root injections of simple Lie algebras of fixed type A, B, C or D.

Consider a specific root injection $\mathfrak{s}_n \subset \mathfrak{s}_{n+1}$. There exists a Weyl group element for $\mathfrak{s}_{n+1}$, understood as an automorphism of $\mathfrak{s}_{n+1}$, that identifies the image of $\mathfrak{s}_n$

with the root subalgebra of $\mathfrak{s}_{n+1}$ whose roots are the same as those of $\mathfrak{s}_n$, i.e., for instance the root space $\mathfrak{s}_n^{\varepsilon_1+\varepsilon_2}$ is mapped exactly into $\mathfrak{s}_{n+1}^{\varepsilon_1+\varepsilon_2}$. This shows that for any fixed type A, B, C or D, all locally simple Lie algebras $\mathfrak{s}$ constructed by root injections of simple Lie algebras of this particular type are isomorphic, as they correspond to the same pair $\{\operatorname{span}\Delta, \Delta\}$ (the condition that $\mathfrak{s}$ is generated by its root spaces is obviously satisfied here). More precisely, this classifies the locally simple infinite-dimensional root-reductive Lie algebras up to a *root isomorphism*, i.e., an isomorphism

$$\varphi : \mathfrak{s} \to \mathfrak{s}'$$

with $\varphi(\mathfrak{h}) = \mathfrak{h}'$ and $\varphi(\mathfrak{s}^\alpha) = \mathfrak{s}^{\alpha'}$ ($\mathfrak{h}$ and $\mathfrak{h}'$ denoting the corresponding unions of finite-dimensional Cartan subalgebras in $\mathfrak{s}$ and $\mathfrak{s}'$ respectively). These finer isomorphism classes are denoted by A_∞, B_∞, C_∞ and D_∞. Less formally, we can think of A_∞, B_∞, C_∞ and D_∞ as Lie algebras with a root decomposition (3.3). The corresponding sets of roots are as follows:

$$\begin{aligned}
A_\infty: &\quad \Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j < \infty\},\\
B_\infty: &\quad \Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid 1 \le i < j < \infty\},\\
C_\infty: &\quad \Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \le i < j < \infty\},\\
D_\infty: &\quad \Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \mid 1 \le i < j < \infty\}.
\end{aligned}$$

As an exercise, the reader should check that no vector space isomorphism of $\operatorname{span}\Delta$ and $\operatorname{span}\Delta'$ sends Δ to Δ', when Δ and Δ' are two different sets of roots from the above list (hint: consider the additive properties of the roots in each system). Consequently, the root isomorphism classes A_∞, B_∞, C_∞ and D_∞ are distinct. It is also important to note that the indices of ε_i in the above infinite root systems can be chosen to belong to any ordered infinite countable set. In the formulas above, the choice of this index set is $\mathbb{Z}_{>0}$, but in some examples it will be convenient to choose it as $\mathbb{Z}$.

A further important observation is that B_∞ and D_∞ are isomorphic as Lie algebras. To see this, denote by V a countable-dimensional vector space with basis $e_1, e_2, e_3, \ldots$ and with bilinear form $(\cdot\,,\cdot) : V \times V \to V$ satisfying $(e_i, e_j) = \delta_{ij}$. Let $V_n = \operatorname{span}\{e_1, \ldots, e_n\}$. Then each V_n is equipped with a nondegenerate symmetric bilinear form, and the inclusions $V_n \subset V_{n+1}$ induce a chain of Lie algebra injections

$$\cdots \longrightarrow \mathfrak{o}(V_n) \xrightarrow{\mu_n} \mathfrak{o}(V_{n+1}) \xrightarrow{\mu_{n+1}} \mathfrak{o}(V_{n+2}) \xrightarrow{\mu_{n+2}} \mathfrak{o}(V_{n+3}) \longrightarrow \cdots . \tag{3.4}$$

Clearly, μ_n is not a root injection (as there is no root injection of $\mathfrak{o}(V_n)$ into $\mathfrak{o}(V_{n+1})$ for an odd n); however it turns out that the composition $\mu_{n+1} \circ \mu_n$ is a root injection for every n. This follows from the fact (which the reader should verify explicitly!) that any embedding of the form $\mathfrak{o}(V_n) \to \mathfrak{o}(V_{n+2})$ is a root injection. Furthermore,

the direct limit of the sequence (3.4) (or, informally, the union of the images of the μ_n's) is isomorphic to the direct limits of the two subsequences

$$\cdots \longrightarrow \mathfrak{o}(V_n) \overset{\mu_{n+1}\circ\mu_n}{\longrightarrow} \mathfrak{o}(V_{n+2}) \overset{\mu_{n+3}\circ\mu_{n+2}}{\longrightarrow} \mathfrak{o}(V_{n+4}) \longrightarrow \cdots \tag{3.5}$$

defined by taking $n = 3$ and $n = 4$. The latter limits are nothing but D_∞ and B_∞. Consequently D_∞ and B_∞ are isomorphic as Lie algebras.

In what follows, we will frequently write $\mathfrak{o}(\infty)$ for the isomorphic Lie algebras B_∞ and D_∞, and $\mathfrak{sp}(\infty)$ for C_∞, and we will refer to $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ as the *classical locally simple Lie algebras*.

It is not difficult to prove that $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ are pairwise non-isomorphic Lie algebras. Here is a proof of the fact that $\mathfrak{sl}(\infty)$ and $\mathfrak{sp}(\infty)$ are not isomorphic; the reader should extend the argument to the pairs $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. Assume that $\mathfrak{sl}(\infty) = \bigcup_n \mathfrak{sl}(n)$, $\mathfrak{sp}(\infty) = \bigcup_n \mathfrak{sp}(2n)$ and fix an $i_0 \in \mathbb{Z}_{>0}$. Then the adjoint representation of $\mathfrak{sl}(\infty)$, considered as an $\mathfrak{sl}(i_0)$-module, equals the union of the adjoint representations of $\mathfrak{sl}(n)$ for $n \geq i_0$, each considered as a $\mathfrak{sl}(i_0)$-module. A more detailed look shows that there is an isomorphism of $\mathfrak{sl}(i_0)$-modules

$$\mathfrak{sl}(\infty) \simeq \mathfrak{sl}(i_0) \oplus (S \otimes V_{i_0}) \oplus (S \otimes V_{i_0}^*) \oplus T, \tag{3.6}$$

where V_{i_0} and $V_{i_0}^*$ are respectively the natural $\mathfrak{sl}(i_0)$-module and its dual, and S and T are the countable-dimensional vector spaces with trivial $\mathfrak{sl}(i_0)$-action.

Similarly, the adjoint $\mathfrak{sp}(\infty)$-module, considered as a $\mathfrak{sp}(2j_0)$-module for some $j_0 \geq 2$, is isomorphic to

$$S^2(V_{2j_0}) \oplus (R \otimes V_{2j_0}) \oplus X, \tag{3.7}$$

where V_{2j_0} is the natural $\mathfrak{sp}(2j_0)$-module, and R and X are infinite-dimensional vector spaces with trivial $\mathfrak{sp}(2j_0)$-action. (Note that $S^2(V_{2j_0})$ is nothing but the adjoint $\mathfrak{sp}(2j_0)$-module.)

Assume now for the sake of a contradiction that $\varphi : \mathfrak{sp}(\infty) \overset{\sim}{\to} \mathfrak{sl}(\infty)$ is an isomorphism of Lie algebras. Without loss of generality, we may assume that $\varphi(\mathfrak{sp}(2j_0)) \subset \mathfrak{sl}(i_0)$ for some i_0, j_0 as above, $i_0 > 2j_0$. Since V_{2j_0} is the only non-trivial simple $\mathfrak{sp}(2j_0)$-constituent of the adjoint representation of $\mathfrak{sp}(\infty)$ with infinite multiplicity, both V_{i_0} and $V_{i_0}^*$ must necessarily be isomorphic to a finite number of copies of V_{2j_0} and a finite number of copies of the one-dimensional trivial $\mathfrak{sp}(2j_0)$-module. Then it is clear that $\Lambda^2(V_{2j_0})$ is a constituent of $\mathfrak{sl}(i_0)$ considered as an adjoint $\mathfrak{sp}(2j_0)$-module, hence (3.6) and (3.7) are necessarily incompatible. This shows that the existence of φ leads to a contradiction. Hence the Lie algebras $\mathfrak{sl}(\infty)$ and $\mathfrak{sp}(\infty)$ are not isomorphic.

3.3 Description of Root-Reductive Lie Algebras

We are now ready to give a general description of root-reductive Lie algebras as established in Theorem 1 in [DP1].

Theorem 3.1 *The following two statements hold for any root-reductive Lie algebra* $\mathfrak{g}$.

(1) Let $\mathfrak{g}'$ *be the Lie subalgebra of* $\mathfrak{g}$ *generated by all root spaces in the decomposition (3.3). Then* $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ *and there is a split exact sequence of Lie algebras*

$$0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{a} = \mathfrak{g}/\mathfrak{g}' \to 0, \tag{3.8}$$

where $\mathfrak{a}$ *is abelian.*

(2) $\mathfrak{g}'$ *is isomorphic to a direct sum of finite-dimensional simple Lie algebras, as well as of copies of* $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ *and* $\mathfrak{sp}(\infty)$ *with at most countable multiplicities.*

Theorem 3.1 provides an explicit description of the subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$, or equivalently of any perfect root-reductive Lie algebra (recall that a Lie algebra is called perfect if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$). Note that although the exact sequence (3.8) always splits, it is in general not true that there is an isomorphism of Lie algebras $\mathfrak{g} \simeq \mathfrak{g}' \oplus \mathfrak{a}$. For instance, as mentioned earlier, $\mathfrak{gl}(\infty) \not\simeq \mathfrak{g}' \oplus \mathbb{C}$ as the center of $\mathfrak{gl}(\infty)$ is trivial, while the Lie algebra $\mathfrak{g}' \oplus \mathbb{C}$ has center isomorphic to $\mathbb{C}$. The problem of giving an explicit description of all isomorphism classes of root-reductive Lie algebras arising from the exact sequence (3.8) is open. Here is an example that shows that this problem is not trivial. Let $\bar{\mathfrak{g}} = \mathfrak{gl}(\infty) \oplus \mathfrak{gl}(\infty)$ and let $\mathfrak{g}$ be the subalgebra in $\bar{\mathfrak{g}}$ singled out by the equation $\operatorname{tr} x_1 - \operatorname{tr} x_2 = 0$ where $x = x_1 \oplus x_2 \in \bar{\mathfrak{g}}$. Then $\mathfrak{g}/\mathfrak{g}' \simeq \mathbb{C}$, however the reader should prove that $\mathfrak{g} \not\simeq \mathfrak{sl}(\infty) \oplus \mathfrak{sl}(\infty) \oplus \mathbb{C}$ and $\mathfrak{g} \not\simeq \mathfrak{gl}(\infty) \oplus \mathfrak{sl}(\infty)$.

Outline of the Proof of Theorem 3.1 Let $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ for fixed reductive Lie algebras $\mathfrak{g}_n$ and root embeddings $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ for fixed Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$. First of all, clearly $[\mathfrak{g}, \mathfrak{g}] = \bigcup_n [\mathfrak{g}_n, \mathfrak{g}_n]$. On the other hand, since every root space $\mathfrak{g}^\alpha$ is a root space of all $\mathfrak{g}_n$ for large enough n, $\mathfrak{g}'$ is contained in the union of Lie subalgebras of $\mathfrak{g}$ generated by the root spaces of $\mathfrak{g}_n$. These latter subalgebras are nothing but $[\mathfrak{g}_n, \mathfrak{g}_n]$, hence $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Therefore $\mathfrak{g}'$ is an ideal and $\mathfrak{g}/\mathfrak{g}'$ is abelian. The fact that the exact sequence (3.8) splits (non-canonically) follows from the fact that any Lie algebra complement $\mathfrak{a}$ of $\mathfrak{h} \cap \mathfrak{g}'$ in $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ that projects isomorphically onto $\mathfrak{a}$, and hence $\mathfrak{g} = \mathfrak{g}' \dotplus \mathfrak{a}$.

To complete the proof it suffices to show that $\mathfrak{g}'$ is a direct sum of locally simple root-reductive Lie algebras. Consider a root space $\mathfrak{g}^\alpha \subset \mathfrak{g}'$. By the definition of root embedding, $\mathfrak{g}^\alpha$ singles out a sequence of simple direct summands $\mathfrak{s}_n(\alpha)$ of $\mathfrak{g}_n$, such that $\mathfrak{g}^\alpha$ is a root space of each $\mathfrak{s}_n(\alpha)$. Let us attach to α the locally simple Lie subalgebra $\mathfrak{s}_\alpha := \bigcup_n \mathfrak{s}_n(\alpha)$ of $\mathfrak{g}'$. Clearly, if $\mathfrak{s}_\alpha = \mathfrak{s}_\beta$ for some root space $\mathfrak{g}^\beta \subset \mathfrak{g}'$, then $\mathfrak{s}_n(\alpha) = \mathfrak{s}_n(\beta)$ for large enough n. The Lie algebras $\mathfrak{s}_\alpha$, for varying

α, are locally simple root-reductive Lie algebras and are thus finite dimensional or isomorphic to $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, or $\mathfrak{sp}(\infty)$ [Ba]. Let R be a set parameterizing all subalgebras $\mathfrak{s}_\alpha$, i.e., let $\{\mathfrak{s}^r \mid r \in R\}$ be the set of distinct subalgebras among the $\mathfrak{s}_\alpha$. Then $\mathfrak{s}^r \cap \mathfrak{s}^{r'} = 0$ for $r \neq r'$, and moreover the reader will check immediately that the sum $\sum_{r\in R} \mathfrak{s}^r$ is direct. Finally, $\sum_{r\in R} \mathfrak{s}^r \subset \mathfrak{g}'$, and in fact $\sum_{r\in R} \mathfrak{s}^r = \mathfrak{g}'$ as $\sum_{r\in R} \mathfrak{s}^r$ contains all root spaces of $\mathfrak{g}$. This proves the theorem. □

To summarize, we now have a description of arbitrary root-reductive Lie algebras, which is explicit modulo a description of the exact sequence (3.8). However note that so far all of our considerations have been based on sequences of root injections and have not been intrinsic in $\mathfrak{g}$.

3.4 Generalized Root Decomposition

In the remainder of this chapter, we will show that an approach based on the existence of an abstract root decomposition also leads to the class of root-reductive Lie algebras. We start with some definitions.

A subalgebra $\mathfrak{k}$ in a locally finite Lie algebra $\mathfrak{g}$ is called *locally solvable* if every finite-dimensional subalgebra of $\mathfrak{k}$ is solvable. The *locally solvable radical* $\mathfrak{r}_\mathfrak{g}$ of an arbitrary locally finite Lie algebra $\mathfrak{g}$ is defined to be the unique maximal locally solvable ideal in $\mathfrak{g}$. (To the reader: please prove that $\mathfrak{r}_\mathfrak{g}$ is well defined by identifying $\mathfrak{r}_\mathfrak{g}$ with the sum of all locally solvable ideals in $\mathfrak{g}$). We call a locally finite Lie algebra $\mathfrak{g}$ *semisimple* if $\mathfrak{r}_\mathfrak{g} = 0$.

Furthermore, for any Lie subalgebra $\mathfrak{k}$ of a locally finite Lie algebra $\mathfrak{g}$, we set

$$\mathfrak{g}^0_\mathfrak{k} := \{x \in \mathfrak{g} \mid \exists N : (\operatorname{ad} k_1 \circ \cdots \circ \operatorname{ad} k_N)(x) = 0 \ \forall k_1, \ldots, k_N \in \mathfrak{k}\}. \tag{3.9}$$

We say that $\mathfrak{k}$ acts *locally nilpotently* on a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ if $\mathfrak{a} \subset \mathfrak{g}^0_\mathfrak{k}$. If $\mathfrak{k}$ acts locally nilpotently on itself (i.e., $\mathfrak{k} \subset \mathfrak{g}^0_\mathfrak{k}$), then it follows that $\mathfrak{k}$ is a *locally nilpotent* subalgebra, i.e., every finite set of elements in $\mathfrak{k}$ generates a nilpotent Lie subalgebra of $\mathfrak{k}$. However the converse does not hold, as we will see in the following example.

Example 3.2 Let $\mathfrak{g} = \mathfrak{sl}(\infty) = \bigcup_n \mathfrak{sl}(n)$, with the inclusions $\mathfrak{sl}(n) \subset \mathfrak{sl}(n+1)$ being upper left (or lower right) corner inclusions. Then any Lie subalgebra $\mathfrak{k}$ of $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$ has the property $\mathfrak{k} \subset \mathfrak{g}^0_\mathfrak{k}$, and moreover if $\mathfrak{k} = \mathfrak{h}$, then we have an equality $\mathfrak{g}^0_\mathfrak{k} = \mathfrak{k}$. On the other hand, $\mathfrak{n} := \bigoplus_{i<j} \mathfrak{g}^{\varepsilon_i - \varepsilon_j}$ is a locally nilpotent Lie subalgebra that does not act locally nilpotently on itself since $\mathfrak{g}^0_\mathfrak{n} = 0$. (Prove this!) ◯

Now let $\mathfrak{g}$ be any locally finite Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra with the property $\mathfrak{h} = \mathfrak{g}^0_\mathfrak{h}$. For any nonzero linear functional α on the quotient $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$, set

$$\mathfrak{g}^\alpha := \left\{ x \in \mathfrak{g} \;\middle|\; \begin{array}{c} \exists N : (\operatorname{ad} h_1 - \alpha(h_1)) \circ \cdots \circ (\operatorname{ad} h_N - \alpha(h_N))(x) = 0, \\ \forall h_1, \ldots, h_N \in \mathfrak{h} \end{array} \right\}.$$

Then the sum of all subspaces $\mathfrak{g}$ of the form $\mathfrak{g}^\alpha$ is direct (to the reader: please prove this), and $\mathfrak{h} \oplus (\bigoplus \mathfrak{g}^\alpha)$ is an $\mathfrak{h}$-submodule of $\mathfrak{g}$. The spaces $\mathfrak{g}^\alpha$ are called the *generalized* $\mathfrak{h}$*-weight spaces* of $\mathfrak{g}$, or *generalized root spaces*. We say that $\mathfrak{g}$ admits a *generalized* $\mathfrak{h}$*-root decomposition* if

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in (\mathfrak{h}/[\mathfrak{h},\mathfrak{h}])^* \setminus \{0\}} \mathfrak{g}^\alpha \right). \tag{3.10}$$

Any root-reductive Lie algebra admits a generalized $\mathfrak{h}$-root decomposition (3.3) for $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$. In addition, $\mathfrak{h}$ is abelian, all generalized root spaces are one-dimensional and those are simply *root spaces*, that is

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid (\operatorname{ad} h - \alpha(h))(x) = 0 \; \forall \, h \in \mathfrak{h}\}.$$

In other words, the generalized root decomposition of a root-reductive Lie algebra has very nice additional properties.

It is natural to ask what class of locally finite Lie algebras admits a generalized root decomposition. This is not a trivial question, as not all locally simple Lie algebras admit a generalized root decomposition. This we will explain in Chap. 4.

3.5 Characterization of Root-Reductive Lie Algebras

We are now ready to give an intrinsic characterization of root-reductive Lie algebras in terms of a generalized root decomposition. The following theorem is the main result of [PStr].

Theorem 3.3 *Any countable-dimensional semisimple locally finite Lie algebra* $\mathfrak{g}$*, that admits a generalized root decomposition with respect to some* $\mathfrak{h}$ *satisfying* $\mathfrak{g}_\mathfrak{h}^0 = \mathfrak{h}$*, is root-reductive. Moreover,* $\mathfrak{h}$ *necessarily equals* $\bigcup_n \mathfrak{h}_n$ *for a choice of root injections* $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ *of reductive finite-dimensional Lie algebras* $\mathfrak{g}_n$ *with fixed Cartan subalgebras* $\mathfrak{h}_n$*.*

It is an immediate corollary of Theorem 3.3, that the semisimplicity of $\mathfrak{g}$ together with the existence of a generalized root decomposition implies that all generalized root spaces $\mathfrak{g}^\alpha$ are simply root spaces and that $\mathfrak{h}$ is abelian. Moreover, since $\mathfrak{g}$ is root-reductive, statements *(1)* and *(2)* of Theorem 3.1 apply to $\mathfrak{g}$.

Note however that a root-reductive Lie algebra is not necessarily semisimple, for instance $\mathfrak{r}_\mathfrak{g} = \mathbb{C}$ for $\mathfrak{g} = \mathfrak{sl}(\infty) \oplus \mathbb{C}$. A root-reductive Lie algebra $\mathfrak{g}$ is semisimple if and only if its center $Z_\mathfrak{g}$ equals zero. (Prove that the radical $\mathfrak{r}_\mathfrak{g}$ of a root-reductive Lie algebra equals its center!)

It is easy to single out all semisimple root-reductive Lie algebras in terms of the exact sequence (3.8). Namely, those are exactly all $\mathfrak{g}$ for which the sequence (3.8) has the property that if $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ is any subalgebra whose preimage $\tilde{\mathfrak{g}}$ in $\mathfrak{g}$ is isomorphic

to $\mathfrak{g}' \oplus \tilde{\mathfrak{a}}$, then $\tilde{\mathfrak{a}}$ equals zero. Equivalently, this means that $\mathfrak{g}'$ is a faithful $\mathfrak{a}$-module with respect to the $\mathfrak{a}$-module structure on $\mathfrak{g}'$ induced by the exact sequence (3.8). As an exercise, the reader can supply the proof.

Outline of the Proof of Theorem 3.3

Step 1 We first claim that if $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ is an arbitrary exhaustion by finite-dimensional Lie algebras of a locally finite Lie algebra $\mathfrak{g}$ with $\mathfrak{r}_\mathfrak{g} = 0$, then for any n there exists $j(n) > n$ such that $\mathfrak{r}_{\mathfrak{g}_n} \cap \mathfrak{r}_{\mathfrak{g}_{j(n)}} = 0$. This is an important general observation due to Yu. Bahturin and H. Strade [BS]. Indeed, note first that $\mathfrak{r}_{\mathfrak{g}_j} \cap \mathfrak{g}_n \subset \mathfrak{r}_{\mathfrak{g}_n}$ for any $j > n$, since $\mathfrak{r}_{\mathfrak{g}_n}$ is the maximal solvable ideal in $\mathfrak{g}_n$. Thus $\mathfrak{r}_{\mathfrak{g}_{j'}} \cap \mathfrak{r}_{\mathfrak{g}_n} \subset \mathfrak{r}_{\mathfrak{g}_j} \cap \mathfrak{r}_{\mathfrak{g}_n}$ for any $j' > j > n$ since $\mathfrak{r}_{\mathfrak{g}_{j'}} \cap \mathfrak{g}_j \subset \mathfrak{r}_{\mathfrak{g}_j}$. On the other hand we have $\bigcap_{j>n} \mathfrak{r}_{\mathfrak{g}_j} \subset \mathfrak{r}_\mathfrak{g}$, and since $\mathfrak{r}_\mathfrak{g} = 0$, we also have $\bigcap_{j>n} (\mathfrak{r}_{\mathfrak{g}_n} \cap \mathfrak{r}_{\mathfrak{g}_j}) \subset \bigcap_{j>n} \mathfrak{r}_{\mathfrak{g}_j} = 0$. This, together with the finite-dimensionality of $\mathfrak{r}_{\mathfrak{g}_n}$, implies $\mathfrak{r}_{\mathfrak{g}_n} \cap \mathfrak{r}_{\mathfrak{g}_{j(n)}} = 0$ for some $j(n) > n$.

Step 2 We fix a generalized root decomposition of $\mathfrak{g}$, and will now choose an exhaustion $\bigcup_n \mathfrak{g}_n$ of $\mathfrak{g}$ that is compatible with this generalized root decomposition. More precisely, let $x_1, x_2, \ldots$ be an ordered basis of $\mathfrak{g}$ such that each x_j belongs to $\mathfrak{h}$ or to a generalized root space α_t of $\mathfrak{g}$. Let $\tilde{\tilde{\mathfrak{g}}}_n$ be the Lie subalgebra of $\mathfrak{g}$ generated by $x_1, \ldots, x_n$. Fix a chain of subspaces $\tilde{\mathfrak{h}}_n$ in $\mathfrak{h}$ so that all linear functions that arise as generalized roots α corresponding to vectors $x_j \in \tilde{\tilde{\mathfrak{g}}}_n$, are distinct when restricted to $\tilde{\mathfrak{h}}_n$. Then define $\tilde{\mathfrak{g}}_n$ to be the Lie subalgebra of $\mathfrak{g}$ generated by $\tilde{\tilde{\mathfrak{g}}}_n$ and $\tilde{\mathfrak{h}}_n$. The reader should check that $\mathfrak{h} \cap \tilde{\mathfrak{g}}_n$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}_n$ and that for any α, $\mathfrak{g}^\alpha \cap \tilde{\mathfrak{g}}_n$ is a generalized root space of $\tilde{\mathfrak{g}}_n$ when $\mathfrak{g}^\alpha \cap \tilde{\mathfrak{g}}_n \neq 0$.

Step 3 Since $\mathfrak{r}_\mathfrak{g} = 0$, using Step 1 we choose a subchain $\tilde{\mathfrak{g}}_{n_i}$ of the chain

$$\cdots \subset \tilde{\mathfrak{g}}_n \subset \tilde{\mathfrak{g}}_{n+1} \subset \cdots$$

such that $\mathfrak{r}_{\mathfrak{g}_{n_i}} \cap \mathfrak{r}_{\mathfrak{g}_{n_{i+1}}} = 0$ for all n. Clearly $\mathfrak{g} = \bigcup_i \tilde{\mathfrak{g}}_{n_i}$.

Step 4 Consider now the injections

$$\tilde{\mathfrak{g}}_{n_i} \to \tilde{\mathfrak{g}}_{n_{i+1}} / \mathfrak{r}_{\tilde{\mathfrak{g}}_{n_{i+1}}}$$

for all i. The construction of $\tilde{\mathfrak{g}}_{n_i}$ implies that those are root injections. However, $\tilde{\mathfrak{g}}_{n_{i+1}} / \mathfrak{r}_{\tilde{\mathfrak{g}}_{n_{i+1}}}$ is a semisimple finite-dimensional Lie algebra and hence the image of $\mathfrak{h} \cap \tilde{\mathfrak{g}}_{n_{i+1}}$ in $\tilde{\mathfrak{g}}_{n_{i+1}} / \mathfrak{r}_{\tilde{\mathfrak{g}}_{n_{i+1}}}$ is abelian and all root spaces of $\tilde{\mathfrak{g}}_{n_{i+1}} / \mathfrak{r}_{\tilde{\mathfrak{g}}_{n_{i+1}}}$ are one-dimensional. Therefore $\mathfrak{h} \cap \tilde{\mathfrak{g}}_{n_i}$ is abelian, and the generalized root spaces of $\tilde{\mathfrak{g}}_{n_i}$ are one-dimensional. Hence $\mathfrak{h}$ is abelian, $\dim \mathfrak{g}^\alpha = 1$ for any root space, and moreover, for each root α, the linear function $-\alpha$ is also a root, and $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ generate an $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$.

Step 5 We are now able to choose an exhaustion of $\mathfrak{g}$ by finite-dimensional reductive Lie algebras $\mathfrak{g}_i$ such that the injections $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ are root injections.

To do this, it suffices to modify our choice of the subalgebras $\tilde{\mathfrak{g}}_n$ by letting $\mathfrak{g}'_n$ be the minimal semisimple Lie algebra containing all vectors $x_1, \ldots, x_n$. Then let $\mathfrak{g}_n$ be the Lie subalgebra generated by $\mathfrak{g}'_n$ and by $\mathfrak{h}_n$, where $\bigcup_n \mathfrak{h}_n$ is an arbitrary fixed exhaustion of the abelian Lie algebra $\mathfrak{h}$ by finite-dimensional subalgebras. This completes the proof.

□

Here are a few final remarks. It is easy now to complete the proof of the fact that a Lie algebra $\mathfrak{g}$ that equals a union $\bigcup \mathfrak{sl}(n)$ under arbitrary Lie algebra inclusions $\nu_n : \mathfrak{sl}(n) \hookrightarrow \mathfrak{sl}(n+1)$ is isomorphic to $\mathfrak{sl}(\infty)$. Indeed, by induction on n, we can build a sequence of Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{sl}(n)$ with $\nu_n(\mathfrak{h}_n) \subset \mathfrak{h}_{n+1}$. For this it suffices to notice that for any maximal toral subalgebra $\mathfrak{h}_n \subset \mathfrak{sl}(n)$, its image $\nu_n(\mathfrak{h}_n)$ is a toral subalgebra of $\mathfrak{sl}(n+1)$, and can thus be extended to a maximal toral subalgebra $\mathfrak{h}_{n+1}$ in $\mathfrak{sl}(n+1)$. (This is a very general remark which is not sensitive to the fact that we consider inclusions $\mathfrak{sl}(n) \subset \mathfrak{sl}(n+1)$). The crucial observation now is that since the adjoint representation of $\mathfrak{sl}(n)$ occurs with multiplicity 1 in $\mathfrak{sl}(n+1)$ considered as an $\mathfrak{sl}(n)$-module via ν_n, the inclusion ν_n is a root injection for any compatible choice of Cartan subalgebras $\mathfrak{h}_n, \mathfrak{h}_{n+1}$ with $\nu_n(\mathfrak{h}_n) \subset \mathfrak{h}_{n+1}$. Consequently, $\mathfrak{g}$ is a locally simple root-reductive Lie algebra and its root system is isomorphic to the root system of A_∞. This implies that $\mathfrak{g}$ is itself isomorphic to $\mathfrak{sl}(\infty)$.

As an exercise the reader could prove that it is not true that a union $\bigcup_n \mathfrak{gl}(n)$ (for some inclusions $\mathfrak{gl}(n) \subset \mathfrak{gl}(n+1)$) is always isomorphic to $\mathfrak{gl}(\infty)$. It can be isomorphic to $\mathfrak{sl}(\infty)$. Can it be non-isomorphic to both $\mathfrak{gl}(\infty)$ and $\mathfrak{sl}(\infty)$? In fact, the reader is urged to try to construct an example of a Lie algebra obtained as a union $\bigcup_n \mathfrak{gl}(n)$ that cannot be embedded into $\mathfrak{gl}(\infty)$!

Note that it is a corollary of both theorems in this chapter that any generalized root decomposition of an infinite-dimensional locally simple and locally finite Lie algebra corresponds to a pair $\{\operatorname{span}\Delta, \Delta\}$, where Δ is exactly one of the root systems A_∞, B_∞, C_∞, and D_∞. For the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{sp}(\infty)$, this assignment is unique, namely, A_∞, C_∞, respectively. However as we indicated earlier, the Lie algebra $\mathfrak{o}(\infty)$ can be represented both as B_∞ and as D_∞; hence, the isomorphism class of a root-reductive Lie algebra does not determine the pair $\{\operatorname{span}\Delta, \Delta\}$ up to isomorphism.

Chapter 4
Two Generalizations

In this chapter we describe a superalgebra analog of the class of root-reductive Lie algebras introduced in the previous chapter. We then introduce a class of locally simple Lie algebras, which is more general than the class of locally simple root-reductive Lie algebras.

4.1 Root-Reductive Lie Superalgebras

4.1.1 Definitions

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a countable-dimensional locally finite Lie superalgebra. Equivalently, $\mathfrak{g}$ is isomorphic to a union

$$\bigcup_{n \in \mathbb{Z}_{>0}} \mathfrak{g}_n$$

of nested finite-dimensional Lie superalgebras $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$. Note that the subscript n of $\mathfrak{g}_n$ should not be confused with the parity notation $\bar{0}$, $\bar{1}$.

Let $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}} \subset \mathfrak{g}$ be a locally nilpotent Lie subsuperalgebra satisfying the condition $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}_{\bar{0}}}$, where (as in Sect. 2.5)

$$\mathfrak{g}^0_{\mathfrak{h}_{\bar{0}}} := \{x \in \mathfrak{g} \mid \exists N : (\operatorname{ad} h_1 \circ \cdots \circ \operatorname{ad} h_N)(x) = 0 \ \forall h_1, \ldots, h_N \in \mathfrak{h}_{\bar{0}}\}.$$

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_4

We say that $\mathfrak{g}$ admits a *generalized* $\mathfrak{h}$*-root decomposition* if $\mathfrak{g}$ admits a ($\mathbb{Z}_2$-graded) vector space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\substack{\alpha \in (\mathfrak{h}_{\bar{0}}/[\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}])^* \\ \alpha \neq 0}} \mathfrak{g}^\alpha \Big),$$

into generalized root spaces

$$\mathfrak{g}^\alpha := \left\{ x \in \mathfrak{g} \;\middle|\; \begin{array}{c} \exists N : ((\operatorname{ad} h_1 - \alpha(h_1)) \circ \cdots \circ (\operatorname{ad} h_N - \alpha(h_N))(x) = 0, \\ \forall h_1, \ldots, h_N \in \mathfrak{h}_{\bar{0}} \end{array} \right\}.$$

The set of all nonzero linear functions $\alpha \in (\mathfrak{h}_{\bar{0}}/[\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}])^*$ with $\mathfrak{g}^\alpha \neq 0$ is by definition the set of *roots* of $\mathfrak{g}$, and we will denote it by Δ.

The above definitions are direct extensions of the definitions from Chap. 3, and one may ask whether Theorems 3.1 and 3.3 generalize to Lie superalgebras. In particular, one would like to have an explicit description of all simple countable-dimensional locally finite Lie superalgebras that admit a generalized root decomposition. As a first step toward accomplishing this goal, one needs a good supply of Lie superalgebras admitting generalized root decompositions. A natural way to construct such Lie superalgebras is as unions of simple Lie superalgebras under root injections. (The definition of a root injection extends in an obvious way to Lie superalgebras.) This leads to a more limited but still rather general problem.

If $\mathfrak{s}$ is an infinite-dimensional Lie superalgebra constructed as the union of simple Lie superalgebras under root injections, then as in the Lie algebra case, at least one series of classical or Cartan type will appear infinitely many times. Therefore it is enough to study unions of simple Lie superalgebras of a fixed classical or Cartan type. The characterization of all locally simple Lie superalgebras admitting generalized root decompositions is currently an open problem, as it has only been solved in the classical case, whereas a solution in the Cartan-type case would be of interest too.

In fact, the above mentioned partial solution to the characterization of locally simple Lie superalgebras leads to a full solution of the original problem, subject to some additional natural conditions. In this chapter we will outline these results. For more details, the reader is referred to the paper [P4].

4.1.2 Classically Semisimple Lie Superalgebras

We now discuss countable-dimensional locally finite Lie superalgebras $\mathfrak{g}$ that are semisimple, where analogous to the Lie algebra case, $\mathfrak{g}$ is called *semisimple* if the locally solvable radical $\mathfrak{r}_\mathfrak{g}$ equals zero (the locally solvable radical of a locally finite Lie superalgebra is by definition the unique maximal locally solvable ideal).

Consider more specifically the class of semisimple countable-dimensional locally finite Lie superalgebras $\mathfrak{g}$ that admit a generalized root decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha)$ that satisfies the following two natural conditions:

(C1) the inherited root decomposition on the even part $\mathfrak{g}_{\bar{0}} = \mathfrak{h}_{\bar{0}} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha_{\bar{0}})$ makes $\mathfrak{g}_{\bar{0}}$ root-reductive;
(C2) $\mathfrak{g}$ is generated by its generalized root spaces $\mathfrak{g}^\alpha$.

We call such Lie superalgebras the *classically semisimple* (locally finite) Lie superalgebras.

The main result of [P4] states that any classically semisimple Lie superalgebra $\mathfrak{g}$ is isomorphic to a direct sum of classical locally simple Lie superalgebras. Before we introduce these simple Lie superalgebras, let us first discuss this result in the finite-dimensional case.

If $\mathfrak{g}$ is finite dimensional, we know that $\mathfrak{g}$ being semisimple is not sufficient to imply that $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie superalgebras. Note, however, that any finite-dimensional Lie superalgebra admits a generalized root decomposition. (To the reader: please prove that any Cartan subsuperalgebra of $\mathfrak{g}$ induces a generalized root decomposition on $\mathfrak{g}$.) Example 4.1 below shows that semisimplicity together with the assumption that $\mathfrak{g}$ is generated by its root spaces (condition (C2)) is not sufficient to ensure that $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie superalgebras. In fact, already in the finite-dimensional case one needs both conditions in the definition of a classically semisimple Lie superalgebra $\mathfrak{g}$ to ensure that $\mathfrak{g}$ is isomorphic to a direct sum of classical or exceptional simple Lie superalgebras.

Example 4.1 Let $\mathfrak{g} = \mathfrak{sl}(2) \otimes \Lambda_n \Subset \mathbf{W}(n)$ for some fixed $n \geq 3$. Then $\mathfrak{r}_\mathfrak{g} = 0$, as seen in Theorem 2.6. Furthermore, if $\mathfrak{h}_{\mathfrak{sl}(2)}$ is a Cartan subalgebra of $\mathfrak{sl}(2)$, then $\mathfrak{h} := (\mathfrak{h}_{\mathfrak{sl}(2)} \otimes \mathbb{C}) \Subset \operatorname{span}(\xi_1 \frac{\partial}{\partial \xi_1}, \ldots, \xi_n \frac{\partial}{\partial \xi_n})$ is a Cartan subsuperalgebra of $\mathfrak{g}$. (Check!) The Lie superalgebra $\mathfrak{g}$ satisfies condition (C2) as the $\mathfrak{h}$-root spaces of $\mathfrak{g}$ generate $\mathfrak{g}$, since one may check that they generate $\mathfrak{h}$. On the other hand, $\mathfrak{g}$ does not satisfy condition (C1), as $\mathfrak{g}_{\bar{0}}$ is obviously a non-reductive Lie algebra. (Please compute $\mathfrak{r}_{\mathfrak{g}_{\bar{0}}}$ as an exercise.)

We observe now that $\mathfrak{g}$ is not isomorphic to a direct sum of simple Lie superalgebras. It suffices to show that the ideal $\mathfrak{sl}(2) \otimes \Lambda_n$ is not a direct sum of simple Lie superalgebras. (Indeed, prove that if $\mathfrak{g}$ is a Lie superalgebra that is isomorphic to a direct sum of simple Lie superalgebras, then every ideal in $\mathfrak{g}$ is also isomorphic to a direct sum of simple Lie superalgebras.) However, this is clear as the radical of $\mathfrak{sl}(2) \otimes \Lambda_n$ is $\mathfrak{sl}(2) \otimes \Lambda_n^+$, where Λ_n^+ is the subalgebra of Λ_n generated by ξ_i for $i = 1, \ldots, n$. ◯

4.1.3 The Finite-Dimensional Case

We now outline the proof of the main result of [P4] in the special case when $\mathfrak{g}$ is finite dimensional. This case is not fully representative of the general case, but it is still quite instructive. We show that if $\mathfrak{g}$ is a finite-dimensional classically semisimple Lie superalgebra, then $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie superalgebras that are of classical or exceptional type, thus motivating the term "classically semisimple."

Theorem 4.2 *Suppose that $\mathfrak{g}$ is a finite-dimensional Lie superalgebra that is classically semisimple. Then $\mathfrak{g} \simeq \bigoplus_k \mathfrak{s}_k$, where $\mathfrak{s}_k$ are simple Lie superalgebras of classical or exceptional type.*

Proof By the Kac–Cheng Theorem (Theorem 2.6), we have

$$\bigoplus_k (\mathfrak{s}_k \otimes \Lambda_{n_k}) \dotplus \mathfrak{d} \subset \mathfrak{g} \subset \bigoplus_k (\operatorname{der} \mathfrak{s}_k \otimes \Lambda_{n_k}) \dotplus \mathfrak{d}$$

for some simple Lie superalgebras $\mathfrak{s}_k$, some integers n_k and some Lie subsuperalgebra $\mathfrak{d} \subset \bigoplus_k \mathbf{W}(n_k)$. It is easy to see that $(\mathfrak{s}_k)_{\bar{0}} \otimes \Lambda^2_{n_k} \subset \mathfrak{r}_{\mathfrak{g}_{\bar{0}}}$ and $(\mathfrak{s}_k)_{\bar{1}} \otimes \Lambda^1_{n_k} \subset \mathfrak{r}_{\mathfrak{g}_{\bar{0}}}$ for any k. Furthermore, neither subspace belongs to the center of $\mathfrak{g}_{\bar{0}}$ unless it is trivial. (Prove!) Therefore since $\mathfrak{g}_{\bar{0}}$ is reductive (condition (C1)), for every k we have $n_k \leq 1$, and $n_k = 0$ for $(\mathfrak{s}_k)_{\bar{1}} \neq 0$. This implies that $\mathfrak{d}$ is isomorphic to a subsuperalgebra of a direct sum of copies of $\mathbf{W}(1)$, and an immediate verification shows that the condition that $\mathfrak{g}$ be generated by its root spaces (condition (C2)) forces $\mathfrak{d}$ to equal zero. Hence we have

$$\bigoplus_k \mathfrak{s}_k \subset \mathfrak{g} \subset \bigoplus_k \operatorname{der} \mathfrak{s}_k. \tag{4.1}$$

Condition (C1) implies that the Lie superalgebras $\mathfrak{s}_k$ cannot be of Cartan type, since otherwise $\mathfrak{g}_{\bar{0}}$ would have a non-reductive ideal $(\mathfrak{s}_{k_0})_{\bar{0}}$. Furthermore, in Proposition 5.1.2 of [K2], there is a list of the Lie superalgebras $\operatorname{der}\mathfrak{s}$ for all finite-dimensional simple Lie superalgebras $\mathfrak{s}$. According to this list, for a classical or exceptional simple Lie superalgebra $\mathfrak{s}$, $\operatorname{der}\mathfrak{s} \neq \mathfrak{s}$ implies that $\mathfrak{s} \simeq \mathfrak{psl}(m|m)$, $\mathfrak{s} \simeq \mathfrak{psq}(m)$ or $\mathfrak{s} \simeq \mathfrak{sp}(m)$, and in these cases $\operatorname{der}\mathfrak{s}$ equals respectively $\mathfrak{pgl}(m|m)$, $\mathfrak{pq}(m)$ or $\mathbf{p}(m)$. None of the Lie superalgebras $\mathfrak{pgl}(m|m)$, $\mathfrak{pq}(m)$ or $\mathbf{p}(n)$ are generated by their root spaces, since the roots spaces of $\operatorname{der}\mathfrak{s}$ generate $\mathfrak{s}$. Therefore the assumption that $\mathfrak{g} \not\simeq \bigoplus_k \mathfrak{s}_k$ immediately contradicts condition (C2). Hence $\mathfrak{g} \simeq \bigoplus_k \mathfrak{s}_k$ where $\mathfrak{s}_k$ are simple Lie superalgebras of classical or exceptional type. □

4.1.4 Classical Locally Simple Lie Superalgebras

In order to state the main result for the infinite-dimensional case, we need to introduce the infinite-dimensional *classical locally simple Lie superalgebras*. By definition, these are the Lie superalgebras $\mathfrak{sl}(\infty|n)$, $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(\infty|2k)$, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{osp}(m|\infty)$, $\mathfrak{sp}(\infty)$ and $\mathfrak{sq}(\infty)$. We will not carry out their constructions in detail, but will instead restrict ourselves to a few informal remarks.

Any classical locally simple Lie superalgebra can be obtained as a direct limit of simple classical Lie superalgebras. To construct each of the above Lie superalgebras, one considers an appropriate chain of root injections, for instance, lower right-hand corner inclusions $\mathfrak{sl}(i|n) \hookrightarrow \mathfrak{sl}(i+1|n)$ for $i \to \infty$. The union is by definition the Lie superalgebra $\mathfrak{sl}(\infty|n)$. To define $\mathfrak{sl}(\infty|\infty)$, one considers inclusions $\mathfrak{sl}(i|i) \hookrightarrow \mathfrak{sl}(i+1|i+1)$.

In the case of $\mathfrak{osp}(\infty|2k)$, one has two natural choices for the inclusions $\mathfrak{osp}(i|2k) \hookrightarrow \mathfrak{osp}(i+2|2k)$ by taking i to be either odd or even. The resulting unions are denoted by $B_{\infty|2k}$ and $D_{\infty|2k}$ respectively, and admit no root isomorphism. However, $B_{\infty|2k}$ and $D_{\infty|2k}$ are easily seen to be isomorphic as Lie superalgebras (essentially by the same argument as for B_∞ and D_∞) and the common notation for this isomorphism class of Lie superalgebras is $\mathfrak{osp}(\infty|2k)$. The same remark applies to $\mathfrak{osp}(\infty|\infty)$.

Another important observation is that the Lie superalgebras $\mathfrak{sq}(n)$ for $n \in \mathbb{Z}_{>0}$ are not simple; however, the union $\mathfrak{sq}(\infty) = \bigcup_n \mathfrak{sq}(n)$ is simple. The proof is left as an exercise for the reader. An informal explanation is that the identity matrix does not belong to the union $\bigcup_{n\in\mathbb{Z}_{>0}} \mathfrak{sq}(n)$, and therefore $\mathfrak{sq}(\infty)$ is simple. The situation for $\mathfrak{sl}(\infty|\infty)$ is very similar.

A potential defect of the above ad hoc constructions is that one does not know a priori whether changing the root embeddings involved might produce non-isomorphic Lie superalgebras in the limit. However, as in the Lie algebra case, the answer to this question is no. More precisely, C. Oseledets has established the following result [O].

Let $\mathfrak{s}$ be a countable-dimensional Lie superalgebra defined via root injections of classical finite-dimensional Lie superalgebras. Then if $\mathfrak{s}$ is simple, $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(\infty|n)$ for some n, $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(\infty|2k)$ for some k, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{osp}(m|\infty)$ for some m, $\mathfrak{sp}(\infty)$, or $\mathfrak{sq}(\infty)$.

The proof is purely combinatorial and is based on the same idea as in the Lie algebra case: if two Lie superalgebras constructed as unions of classical Lie superalgebras via root injections have the "same" root decompositions, they are isomorphic. The reader can already get a feeling for the relevant combinatorics by proving the result of Oseledets under the simplifying assumption that the finite-dimensional Lie superalgebras involved are isomorphic to $\mathfrak{sl}(m|n)$ for varying m and n.

4.1.5 Description of Classically Semisimple Lie Superalgebras

A precise statement of the main result of [P4] is now the following theorem.

Theorem 4.3 *Any classically semisimple locally finite Lie superalgebra is isomorphic to a countable direct sum of copies of* $\mathfrak{sl}(\infty|n)$, $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(\infty|2k)$, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{osp}(m|\infty)$, $\mathfrak{sp}(\infty)$, $\mathfrak{sq}(\infty)$ *and finite-dimensional simple Lie superalgebras of classical and exceptional type.*

Proof *(**Outline of the Proof**)* Similar to the proof of Theorem 3.3 for root-reductive Lie algebras, the first step is to choose an exhaustion of $\mathfrak{g}$

$$\tilde{\mathfrak{g}}_1 \subset \tilde{\mathfrak{g}}_2 \subset \cdots \subset \mathfrak{g} = \bigcup_n \tilde{\mathfrak{g}}_n$$

that is compatible with the generalized root decomposition. One then applies the principle of Bahturin–Strade (whose proof for Lie superalgebras is the same as for Lie algebras) that there exists a subexhaustion $\tilde{\mathfrak{g}}_{n_i}$ with the property that $\mathfrak{r}_{\tilde{\mathfrak{g}}_{n_{i+1}}} \cap \tilde{\mathfrak{g}}_{n_i} = 0$ for all i (see the outline of the proof of Theorem 3.3). This enables one to conclude that $(\tilde{\mathfrak{g}}_{n_i})_{\overline{0}}$ has one-dimensional root spaces, as well as to apply the Kac–Cheng Theorem to the semisimple Lie superalgebras $\tilde{\mathfrak{g}}_{n_i}/\mathfrak{r}_{\tilde{\mathfrak{g}}_{n_i}}$.

The core of the proof is a series of arguments (using all the conditions on $\mathfrak{g}$) that makes it possible to modify the exhaustion $\tilde{\mathfrak{g}}_{n_i}$ to an exhaustion

$$0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g} = \bigcup_n \mathfrak{g}_n,$$

such that each $\mathfrak{g}_n$ is of the form

$$\bigoplus_j \mathfrak{s}_{n_j} \subset \mathfrak{g}_n \subset \bigoplus_j \operatorname{der} \mathfrak{s}_{n_j}$$

for some simple Lie superalgebras $\mathfrak{s}_{nj}$ of classical or Cartan type, and each inclusion $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ is a root injection. The result then follows by using arguments similar to the finite-dimensional case and applying the result of Oseledets [O]. More details can be found in [P4] and in the references therein. □

Before trying to extend Theorem 4.3 to more general locally finite Lie superalgebras with root decomposition, one should study unions of Cartan-type simple Lie superalgebras under root injections. No such study has yet appeared in the literature, but one can show that the obvious analog of the result of Oseledets is false in general. In particular, it fails for root injections of $W(n_i)$ into $W(n_{i+1})$ for $n_i \to \infty$, $i \to \infty$. The reason is that the Lie superalgebras of the form $\bigcup_{n\in\mathbb{Z}_{>0}} W(n)$ obtained by varying the root injections $W(n) \subset W(n+1)$ are not isomorphic. This can be shown using the fact that a union $\bigcup_{i\in\mathbb{Z}_{>0}} \mathfrak{gl}(n_i)$ is not necessarily isomorphic to $\mathfrak{gl}(\infty)$ (see Chap. 3). Nevertheless it is very likely that

one can find an appropriate generalization of Theorem 4.3, or at least classify all locally simple Lie superalgebras that are obtained through chains of root injections.

4.2 Locally Simple Lie Algebras

In the second part of this chapter, we discuss briefly the class of locally simple, locally finite Lie algebras which are not root-reductive.

4.2.1 Diagonal Locally Simple Lie Algebras

Let $\mathfrak{s} = \bigcup_{n\in\mathbb{Z}_{>0}} \mathfrak{s}_n$, where

$$\mathfrak{s}_1 \subset \mathfrak{s}_2 \cdots \subset \mathfrak{s}_n \subset \mathfrak{s}_{n+1} \subset \cdots \tag{4.2}$$

is a non-stabilizing chain of simple Lie algebras. Surprisingly little is known in general about the isomorphism classes of Lie algebras that are unions of arbitrary chains (4.2). Obviously, if $\mathfrak{s}$ is infinite dimensional, one of the types A, B, C, D must appear infinitely often, so one can reduce to the case that all $\mathfrak{s}_k$ are of the same classical type A, B, C or D. Of course, one has to address separately the problem of when two Lie algebras $\mathfrak{s}$ obtained as unions of simple Lie algebras $\mathfrak{s}_i$ of two different types are isomorphic (like B_∞ and D_∞ in the root-reductive case).

Here is the first example of an inclusion $\mathfrak{s}_1 \subset \mathfrak{s}_2$ that is not a root injection. Let $\mathfrak{s}_1$ be isomorphic to A_{n-1}, B_n, C_n or D_n, and let $\mathfrak{s}_2$ be isomorphic to A_{mn-1}, B_{mn}, C_{mn} or D_{mn}, respectively (so that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are of the same type), and consider the inclusion $\mathfrak{s}_1 \subset \mathfrak{s}_2$

$$\varphi_m : \mathfrak{s}_1 \hookrightarrow \mathfrak{s}_2,$$
$$S \longmapsto \begin{pmatrix} S & & 0 \\ & S & \\ & & \ddots \\ 0 & & S \end{pmatrix},$$

where the matrix S is an element of $\mathfrak{s}_1$ and it is repeated m times on the diagonal of the matrix $\varphi_m(S) \in \mathfrak{s}_2$. We call φ_m the *m-fold diagonal inclusion*.

An intrinsic and more general definition of a *diagonal inclusion* is an inclusion $\mathfrak{s}_1 \subset \mathfrak{s}_2$ of classical simple Lie algebras such that the natural representation of $\mathfrak{s}_2$ considered as an $\mathfrak{s}_1$-module is isomorphic to a direct sum of copies of the natural $\mathfrak{s}_1$-module, its dual, and the trivial one-dimensional $\mathfrak{s}_1$-module. This definition is due to A. Baranov and A. Zhilinski, [BaZ], and in particular, a root inclusion of classical

simple Lie algebras is a special case of a diagonal inclusion. Also, the canonical inclusions $\mathfrak{o}(n) \subset \mathfrak{sl}(n)$ and $\mathfrak{sp}(2n) \subset \mathfrak{sl}(2n)$ are clearly diagonal.

Here is an example of a diagonal inclusion that involves both the natural representation of $\mathfrak{sl}(i)$ and its dual:

$$\begin{aligned} \mathfrak{sl}(i) &\hookrightarrow \mathfrak{sl}(2i+1), \\ S &\longmapsto \begin{pmatrix} S & 0 & 0 \\ 0 & -S^t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

By definition, we call a locally simple locally finite Lie algebra $\mathfrak{s}$ *diagonal* if it admits an exhaustion $\mathfrak{s} = \bigcup_{n \in \mathbb{Z}_{>0}} \mathfrak{s}_n$ where all inclusions $\mathfrak{s}_n \subset \mathfrak{s}_{n+1}$ are diagonal.

4.2.2 Supernatural Numbers

In what follows, we introduce a nice invariant of an arbitrary locally simple, locally finite Lie algebra $\mathfrak{s}$: this invariant is an equivalence class $\mathrm{Stz}(\mathfrak{s})$ of supernatural numbers. We first need some definitions. The multiplicative monoid of *supernatural numbers*, denoted by Stz, consists of all expressions of the form $2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}13^{a_6}\cdots$ where $a_i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, i.e., of all infinite products of primes raised to finite or infinite powers. Supernatural numbers are also known as *Steinitz numbers*.

The (commutative) multiplication is defined in the obvious way, i.e., via addition of the respective powers. $\mathbb{Z}_{>0}$ is the submonoid of Stz, consisting of all monomials for which all powers of primes are finite and at most finitely many are nonzero. Moreover, any infinite product of nonzero natural numbers determines a well-defined supernatural number. We call two supernatural numbers S_1 and S_2 *equivalent* if $mS_1 = nS_2$ for some $m, n \in \mathbb{Z}_{>0}$.

Next we need to recall E. Dynkin's notion of the index of a subalgebra [Dyn]. Let $\mathfrak{s}_1 \subset \mathfrak{s}_2$ be an inclusion of simple finite-dimensional Lie algebras. We can assume that the Killing forms of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are normalized so that the length of each respective longest root equals $\sqrt{2}$. The restriction of the Killing form of $\mathfrak{s}_2$ to $\mathfrak{s}_1$ is an invariant bilinear form on $\mathfrak{s}_1$; hence, it is proportional to the Killing form of $\mathfrak{s}_1$. The coefficient of proportionality arising from the so-normalized Killing forms is the *index of* $\mathfrak{s}_1$ *in* $\mathfrak{s}_2$. In what follows, we will denote the index by $I_{\mathfrak{s}_2}\mathfrak{s}_1$. For example, the reader should check directly from the definition of $I_{\mathfrak{s}_2}\mathfrak{s}_1$ that $I_{\mathfrak{s}_2}\varphi_m(\mathfrak{s}_1) = m$ for any simple Lie algebra $\mathfrak{s}_1$ embedded via the m-diagonal inclusion into the simple Lie algebra $\mathfrak{s}_2$.

The index has two crucial properties. The first, and obvious one, is that the index is multiplicative with respect to chains of inclusions $\mathfrak{s}_1 \subset \mathfrak{s}_2 \subset \mathfrak{s}_3$, i.e., $I_{\mathfrak{s}_3}\mathfrak{s}_1 = I_{\mathfrak{s}_2}\mathfrak{s}_1 I_{\mathfrak{s}_3}\mathfrak{s}_2$. The second, and non-obvious one, is that the index is always a positive integer. For the proof of this fact, see [Dyn, Th. 2.2].

We need one more specific fact about the index. Let V be an irreducible $\mathfrak{s}_1$-module of highest weight λ (with respect to a fixed Borel subalgebra $\mathfrak{b} \subset \mathfrak{s}_1$). Then there is a natural injection

$$\mathfrak{s}_1 \to \mathfrak{sl}(V),$$

and Theorem 2.5 in [Dyn] states that

$$I_{\mathfrak{sl}(V)\mathfrak{s}_1} = \frac{\dim V}{\dim \mathfrak{s}_1}(\lambda, \lambda + 2\rho),$$

where 2ρ stands for the sum of positive roots of $\mathfrak{b}$ and $(\cdot\,,\,\cdot)$ stands for the Killing form of $\mathfrak{s}_1$ normalized as above. We strongly recommend the reader to become acquainted with the classical work of Dynkin [Dyn].

Now let $\mathfrak{s}$ be a locally simple locally finite Lie algebra. An exhaustion $\mathfrak{s} = \bigcup_i \mathfrak{s}_i$ of $\mathfrak{s}$ yields the supernatural number $\prod_i I_{\mathfrak{s}_{i+1}\mathfrak{s}_i}$. If we choose another exhaustion of $\mathfrak{s}$, say $\mathfrak{s} = \bigcup_i \mathfrak{s}'_i$, and fix a large enough i_0 so that $\mathfrak{s}_1 \subset \mathfrak{s}'_{i_0}$, then for any large enough j there exists $k(j)$ such that

$$\mathfrak{s}_1 \subset \mathfrak{s}'_{i_0} \subset \mathfrak{s}_j \subset \mathfrak{s}'_{k(j)}.$$

Therefore $I_{\mathfrak{s}_j\mathfrak{s}_1}$ divides $I_{\mathfrak{s}'_{k(j)}\mathfrak{s}_1} = I_{\mathfrak{s}'_{i_0}\mathfrak{s}_1} I_{\mathfrak{s}'_{k(j)}\mathfrak{s}'_{i_0}}$. If we set $I_{\mathfrak{s}'_{i_0}\mathfrak{s}_1} = m_{i_0}$, then $I_{\mathfrak{s}_j\mathfrak{s}_1}$ divides $m_{i_0} I_{\mathfrak{s}'_{k(j)}\mathfrak{s}'_{i_0}} I_{\mathfrak{s}'_{i_0}\mathfrak{s}'_1} = m_{i_0} I_{\mathfrak{s}'_{k(j)}\mathfrak{s}'_1}$. By letting j go to ∞, we obtain that $\prod_i I_{\mathfrak{s}_{i+1}\mathfrak{s}_i}$ divides $m_{i_0} \prod_i I_{\mathfrak{s}'_{i+1}\mathfrak{s}'_i}$. By switching the roles of the exhaustions $\bigcup_i \mathfrak{s}_i$ and $\bigcup_i \mathfrak{s}'_i$, we see that $\prod_i I_{\mathfrak{s}'_{i+1}\mathfrak{s}'_i}$ divides $n_{j_0} \prod_i I_{\mathfrak{s}_{i+1}\mathfrak{s}_i}$ for some $n_{j_0} \in \mathbb{Z}_{>0}$. This implies (prove!) that the supernatural numbers $\prod_i I_{\mathfrak{s}_{i+1}\mathfrak{s}_i}$ and $\prod_i I_{\mathfrak{s}'_{i+1}\mathfrak{s}'_i}$ are equivalent. Therefore to each isomorphism class of locally simple locally finite Lie algebras $\mathfrak{s}$, we have assigned an equivalence class $\mathrm{Stz}(\mathfrak{s})$ of supernatural numbers. An important corollary of this construction is that if $\mathrm{Stz}(\mathfrak{s}) \neq \mathrm{Stz}(\mathfrak{l})$ for some locally simple locally finite Lie algebras $\mathfrak{s}$ and $\mathfrak{l}$, then $\mathfrak{s}$ and $\mathfrak{l}$ are not isomorphic. However, as we will see immediately, $\mathrm{Stz}(\mathfrak{s})$ is only a rough invariant of $\mathfrak{s}$, and it can happen that $\mathrm{Stz}(\mathfrak{s}) = \mathrm{Stz}(\mathfrak{l})$ for non-isomorphic $\mathfrak{s}$ and $\mathfrak{l}$.

We are now ready to look at some examples. Our first observation is that if $\mathfrak{s}_1 \subset \mathfrak{s}_2$ is a root injection of simple finite-dimensional Lie algebras, then $I_{\mathfrak{s}_2\mathfrak{s}_1} = 1$. (Prove this as an exercise!) Now, if $\mathfrak{s}$ is a root-reductive Lie algebra and $\mathfrak{s} = \bigcup_{k\in\mathbb{Z}_{>0}} \mathfrak{s}_k$ is any exhaustion, then for all k sufficiently large, $\mathfrak{s}_k \subset \mathfrak{s}_{k+1}$ is a root injection [DP4]. Therefore, the equivalence class of supernatural numbers assigned to a locally simple root-reductive Lie algebra $\mathfrak{s}$ is equal to the equivalence class $\mathbb{Z}_{>0}$, i.e., $\mathrm{Stz}(\mathfrak{s}) = \mathbb{Z}_{>0}$. This shows in particular that non-isomorphic algebras, such as $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$, can have the same equivalence class of supernatural numbers.

Recall that a diagonal locally simple locally finite Lie algebra is by definition a Lie algebra $\mathfrak{s}$ that equals $\bigcup_i \mathfrak{s}_i$, where all $\mathfrak{s}_i \subset \mathfrak{s}_{i+1}$ are diagonal inclusions of simple finite-dimensional Lie algebras. The paper [BaZ] of Baranov and Zhilinski gives a complete description of all isomorphism classes of diagonal locally finite locally simple Lie algebras. The result is beautiful but somewhat too technical to present here. We urge the interested reader to study the paper [BaZ]. We will restrict our attention to certain examples.

The most basic class of diagonal Lie algebras is obtained by iterating the m-fold diagonal inclusions

$$\varphi_m : \mathfrak{sl}(i) \hookrightarrow \mathfrak{sl}(im)$$

for varying m. If $\theta_1, \theta_2, \ldots$ is an infinite sequence of elements of $\mathbb{Z}_{>0}$, then let $\mathfrak{sl}(i_0; \prod_i \theta_i)$ ($\prod_i \theta_i$ is a supernatural number!) denote the union of $\mathfrak{sl}(i_0; \theta_1\theta_2\cdots\theta_i)$ under the inclusions

$$\varphi_{k_{i+1}} : \mathfrak{sl}(i_0; \theta_1\theta_2\cdots\theta_i) \hookrightarrow \mathfrak{sl}(i_0; \theta_1\theta_2\cdots\theta_{i+1}).$$

The question of when the Lie algebras $\mathfrak{sl}(i_0; \prod_i \theta_i)$ and $\mathfrak{sl}(j_0; \prod_j \eta_j)$ are isomorphic is a particular case of the classification problem solved by Baranov and Zhilinski. The main result of [BaZ] implies that $\mathfrak{sl}(i_0; \prod_i \theta_i)$ and $\mathfrak{sl}(j_0; \prod_j \eta_j)$ are isomorphic if and only if the supernatural numbers $i_0 \prod_i \theta_i$ and $j_0 \prod_j \eta_j$ are equal. Note that this claim is stronger than the statement that the equivalence classes of supernatural numbers corresponding to $\mathfrak{sl}(i_0; \prod_i \theta_i)$ and $\mathfrak{sl}(j_0; \prod_j \eta_j)$ are equal. We recommend the reader to extract the above claim from Theorem 4.1 in [BaZ].

Here is a very specific example. Consider the Lie algebras $\mathfrak{sl}(2; 2^\infty 3^\infty)$, $\mathfrak{sl}(3; 2^\infty 3^\infty)$ and $\mathfrak{sl}(7; 2^\infty 3^\infty)$, defined respectively by the sequences

$$\mathfrak{sl}(2) \overset{\varphi_2}{\hookrightarrow} \mathfrak{sl}(4) \overset{\varphi_3}{\hookrightarrow} \mathfrak{sl}(12) \overset{\varphi_2}{\hookrightarrow} \mathfrak{sl}(24) \overset{\varphi_3}{\hookrightarrow} \mathfrak{sl}(72) \overset{\varphi_2}{\hookrightarrow} \cdots,$$

$$\mathfrak{sl}(3) \overset{\varphi_3}{\hookrightarrow} \mathfrak{sl}(9) \overset{\varphi_2}{\hookrightarrow} \mathfrak{sl}(18) \overset{\varphi_3}{\hookrightarrow} \mathfrak{sl}(54) \overset{\varphi_2}{\hookrightarrow} \mathfrak{sl}(108) \overset{\varphi_3}{\hookrightarrow} \cdots,$$

$$\mathfrak{sl}(7) \overset{\varphi_2}{\hookrightarrow} \mathfrak{sl}(14) \overset{\varphi_3}{\hookrightarrow} \mathfrak{sl}(42) \overset{\varphi_2}{\hookrightarrow} \mathfrak{sl}(84) \overset{\varphi_3}{\hookrightarrow} \mathfrak{sl}(252) \overset{\varphi_2}{\hookrightarrow} \cdots.$$

Then according to the above, $\mathfrak{sl}(2; 2^\infty 3^\infty)$ and $\mathfrak{sl}(3; 2^\infty 3^\infty)$ are isomorphic, and both are non-isomorphic to $\mathfrak{sl}(7; 2^\infty 3^\infty)$. Can you prove this directly? (Hint: use an argument similar to the one used in the proof that $\mathfrak{sl}(\infty)$ and $\mathfrak{sp}(\infty)$ are not isomorphic, see Chap. 3, Sect. 3.2.) Note that all three Lie algebras have the same equivalence class of supernatural numbers.

4.2.3 Non-diagonal Locally Simple Lie Algebras

In the remainder of the chapter, we discuss the most basic examples of non-diagonal locally simple, locally finite Lie algebras. Let $\mathfrak{sl}(i, \mathrm{adj})$ denote the union

$$\mathfrak{sl}(i) \subset \mathfrak{sl}(i^2 - 1) \subset \mathfrak{sl}((i^2 - 1)^2 - 1) \subset \cdots, \tag{4.3}$$

where each inclusion is induced by the adjoint representation of the corresponding subalgebra: $\mathfrak{sl}(i)$ is considered as a subalgebra of the endomorphisms of the (i^2-1)-dimensional vector space V of the adjoint $\mathfrak{sl}(i)$-module, and so on. Dynkin's result about $I_{\mathfrak{sl}(V)\mathfrak{s}_1}$ allows us to easily compute the equivalence class of supernatural numbers associated with $\mathfrak{sl}(i, \mathrm{adj})$. Indeed, consider the injection

$$\mathfrak{sl}(i) \subset \mathfrak{sl}(i^2 - 1).$$

Here $\mathfrak{s}_1 = \mathfrak{sl}(i)$, V is the adjoint $\mathfrak{sl}(i)$-module, and the highest weight of V is $\lambda = \varepsilon_1 - \varepsilon_i$. A straightforward calculation shows that $(\lambda, \lambda + 2\rho) = 2i$, hence $I_{\mathfrak{sl}(i^2-1)\mathfrak{sl}(i)} = \frac{i^2-1}{i^2-1} \cdot 2i = 2i$. The supernatural number determined by the exhaustion (4.3) is

$$(2i)(2(i^2 - 1))(2((i^2 - 1)^2 - 1)) \cdots. \tag{4.4}$$

This supernatural number $2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}13^{a_6}\cdots$ has the special property that $a_k = 0$ or $a_k = \infty$ for each k. For instance, clearly $a_1 = \infty$. More generally, if a prime p divides the k-th term $j(k)$ of the sequence

$$2i,\ 2(i^2 - 1),\ 2((i^2 - 1)^2 - 1),\ \ldots \tag{4.5}$$

then p also divides the $(k + 2)$-th term $j(k + 2)$ of this sequence, as

$$j(k+2) = 2\left(\left(\left(\frac{j(k)}{2}\right)^2 - 1\right)^2 - 1\right) = 2\left(\frac{j(k)}{2}\right)^4 - 2\left(\frac{j(k)}{2}\right)^2.$$

Therefore if p divides one term in (4.5), it divides infinitely many terms. This means that computing the supernatural number (4.4) is equivalent to determining which primes divide (4.4).

Let's think now for which i and j the Lie algebras $\mathfrak{sl}(i, \mathrm{adj})$ and $\mathfrak{sl}(j, \mathrm{adj})$ are isomorphic. Clearly, if j is a term of the sequence (4.5), or if i is a term of the sequence obtained from (4.5) by replacing i with j, we have $\mathfrak{sl}(i, \mathrm{adj}) \simeq \mathfrak{sl}(j, \mathrm{adj})$. A very natural conjecture is that these are necessary and sufficient conditions for $\mathfrak{sl}(i, \mathrm{adj})$ and $\mathfrak{sl}(j, \mathrm{adj})$ to be isomorphic. A computer experiment (carried out by H. Gangl and then repeated by B. Vioreanu) confirms this conjecture for $i, j < 5000$. In particular, for each pair $(i, j), i < j$, such that j does not belong to the

sequence (4.5), the computer program finds an explicit prime p that divides one of the supernatural numbers

$$(2i)(2(i^2-1))(2((i^2-1)^2-1))\cdots,$$
$$(2j)(2(j^2-1))(2((j^2-1)^2-1))\cdots$$

but does not divide the other.

An interesting side question is: what are all primes that divide all supernatural numbers (4.4) for $i \in \mathbb{Z}_{>0}, i \geq 2$? The answer to this question seems to be unknown, but computer experiments suggest at least four such numbers: 2, 3, 7 and 23. As an exercise, the reader can prove that indeed these four numbers divide the expression in (4.4). To summarize, for the class of Lie algebras of the form $\mathfrak{s} = \mathfrak{sl}(i, \mathrm{adj})$, the equivalence class of supernatural numbers $\mathrm{Stz}(\mathfrak{s})$ is an invariant which conjecturally determines $\mathfrak{s}$ up to isomorphism.

Here is a final example. For every $i \geq 4$, define the Lie algebra $\mathfrak{sl}(i, \Lambda^2)$ as the union

$$\mathfrak{sl}(i) \subset \mathfrak{sl}\left(\frac{i(i-1)}{2}\right) \subset \mathfrak{sl}\left(\frac{\frac{i(i-1)}{2}\cdot\left(\frac{i(i-1)}{2}-1\right)}{2}\right) \subset \cdots \tag{4.6}$$

where the inclusion $\mathfrak{sl}(i) \subset \mathfrak{sl}(\frac{i(i-1)}{2})$ is simply the inclusion

$$\mathfrak{sl}(i) \hookrightarrow \mathfrak{sl}(\dim \Lambda^2(V))$$

corresponding to the representation $\Lambda^2(V)$ of $\mathfrak{sl}(i)$, where V is the natural representation. This inclusion is iterated to produce the chain (4.6). The index $I_{\mathfrak{sl}(\Lambda^2(V))}\mathfrak{sl}(i)$ equals $\frac{i(i-1)}{2(i^2-1)}(\lambda, \lambda+\rho)$, where λ is the highest weight of $\Lambda^2(V)$. A direct computation yields $(\lambda, \lambda+\rho) = 2i - 2 - \frac{4}{i}$, and hence $I_{\mathfrak{sl}(\Lambda^2(V))}\mathfrak{sl}(i) = \frac{i(i-1)}{2(i^2-1)} \cdot (2i - 2 - \frac{4}{i}) = \frac{i^2-i-2}{i+1} = i - 2$. The supernatural number corresponding to the exhaustion (4.6) of $\mathfrak{sl}(i, \Lambda^2)$ is thus

$$(i-2)\left(\frac{i(i-1)}{2}-2\right)\left(\frac{\frac{i(i-1)}{2}\left(\frac{i(i-1)}{2}-1\right)}{2}-2\right)\cdots. \tag{4.7}$$

In contrast to the previous case, the supernatural number $2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}\cdots$ satisfies $a_k \in \mathbb{Z}_{\geq 0}$ for all $k > 1$. Indeed, if for some prime $p > 2$, p divides the k-th term of the sequence

$$i-2,\ \frac{i(i-1)}{2}-2,\ \frac{\frac{i(i-1)}{2}\cdot\left(\frac{i(i-1)}{2}-1\right)}{2}-2,\ \ldots$$

then p does not divide any $(k+\ell)$-th term for $\ell > 0$, as the remainder of the $(k+1)$-th term modulo p equals -1, and the remainder of all subsequent terms equals -2. It is a nice elementary problem in number theory to check that $a_1 = \infty$. (Please do it!) The question of when two supernatural numbers of the form (4.7) are equivalent is open. Can you say something about it?

Chapter 5
Splitting Borel Subalgebras of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ and Generalized Flags

Borel subalgebras play a prominent role in the representation theory of finite-dimensional Lie algebras and superalgebras, as well as of Kac–Moody algebras, since Borel subalgebras are responsible for the very existence of highest weight modules. Therefore it is important to find a proper analog of Borel subalgebras for locally finite Lie algebras. This problem still awaits a comprehensive study, as even the case of root-reductive locally finite Lie algebras is rather nontrivial. We will discuss this latter case throughout this and the next chapter.

5.1 Splitting Borel Subalgebras

Recall that there are at least two approaches to Borel subalgebras of a finite-dimensional Lie algebra: as maximal solvable subalgebras, or via subsets $\Delta_{\mathfrak{b}}$ of the root system Δ satisfying the following conditions from Chap. 1:

(B1) $\Delta_{\mathfrak{b}}$ is additive within Δ, i.e., $\alpha \in \Delta_{\mathfrak{b}}$, $\beta \in \Delta_{\mathfrak{b}}$, and $\alpha + \beta \in \Delta$ imply $\alpha + \beta \in \Delta_{\mathfrak{b}}$;
(B2) for any $\alpha \in \Delta$, exactly one of the following possibilities holds: $\alpha \in \Delta_{\mathfrak{b}}$ or $-\alpha \in \Delta_{\mathfrak{b}}$.

For the classical simple Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$ there is a third well-known approach via flags in the natural representation. More precisely, Borel subalgebras of $\mathfrak{sl}(n)$ are exactly the stabilizers of all possible flags of maximal length in the natural (n-dimensional) representation. For $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$, one has to consider isotropic flags of maximal length.

All three approaches make sense also for finitary root-reductive locally finite Lie algebras, i.e., root-reductive Lie algebras that admit an embedding into $\mathfrak{gl}(\infty)$,[1] and

[1] The theory of finitary root-reductive Lie algebras goes back to A. Baranov, [Ba].

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_5

lead to interesting results. First, if $\mathfrak{g}$ is an arbitrary locally finite Lie algebra, one can define a *Borel subalgebra* of $\mathfrak{g}$ as a maximal locally solvable subalgebra of $\mathfrak{g}$. This definition is very general, and the main problem with it is that it is neither clear how to describe all such subalgebras, nor why they are useful. Nevertheless, in all special cases that we understand, this definition turns out to be surprisingly interesting. The other two approaches cannot be applied in this generality as a general locally finite Lie algebra admits no root decomposition and there is no accepted definition of a natural representation. However, if $\mathfrak{g}$ is root-reductive, those approaches make sense too. Furthermore, in the latter case Theorem 3.1 from Chap. 3 allows us to reduce the problem to just the three simple infinite-dimensional root-reductive Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. We will first discuss the case of $\mathfrak{sl}(\infty)$ in more detail.

5.1.1 Splitting Borel Subalgebras of $\mathfrak{sl}(\infty)$

It is a challenge to describe explicitly all maximal locally solvable subalgebras of $\mathfrak{sl}(\infty)$. This was accomplished in [DP3], and we will present the solution briefly in the next chapter. In the meantime we will consider the root approach to Borel subalgebras for $\mathfrak{sl}(\infty)$. Throughout this subsection we let $\mathfrak{g} := \mathfrak{sl}(\infty)$.

Recall from Chap. 3, that if $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$ with $\mathfrak{h} = \mathfrak{g}_{\mathfrak{h}}^0$ and such that $\mathfrak{g}$ admits a generalized $\mathfrak{h}$-root decomposition (3.10), then there is an exhaustion

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \bigcup_i \mathfrak{g}_i = \mathfrak{g},$$

where $\mathfrak{g}_i \simeq \mathfrak{sl}(i)$, $\mathfrak{h} = \bigcup_i \mathfrak{h}_i$ for some fixed Cartan subalgebras $\mathfrak{h}_i \subset \mathfrak{sl}(i)$, and the inclusions $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ are root injections. Furthermore, if V_i is the natural representation of $\mathfrak{sl}(i)$, clearly $\bigcup_i V_i$ is a well-defined representation of $\mathfrak{sl}(\infty)$, which we call the *natural* $\mathfrak{sl}(\infty)$*-module*. Furthermore, the subalgebra $\mathfrak{h} := \bigcup_i \mathfrak{h}_i$ acts diagonalizably on V (i.e., each element $h \in \mathfrak{h}$ is a semisimple operator on V and all operators $h \in \mathfrak{h}$ are simultaneously diagonalizable) with one-dimensional eigenspaces. Let the corresponding eigenvalue functions be $\varepsilon_1, \varepsilon_2, \ldots \in \mathfrak{h}^*$. Then

$$V = \bigoplus_i V^{\varepsilon_i} \tag{5.1}$$

where each V^{ε_i} is a one-dimensional $\mathfrak{h}$-eigenspace. Set

$$V_* := \bigoplus_i \left(V^{\varepsilon_i}\right)^*.$$

By definition, V_* is the direct sum of the one-dimensional spaces dual to V^{ε_i}, and V_* is a countable-dimensional subspace of the uncountable-dimensional space V^*. The subspace V_* depends on the decomposition (5.1). (Check this as an

exercise!) Clearly, $(V^{\varepsilon_i})^*$ is an $\mathfrak{h}$-eigenspace with eigenvalue function $-\varepsilon_i$. Hence $V_* = \bigoplus_i V_*^{-\varepsilon_i}$.

Consider now the tensor product $V \otimes V_*$ which is an associative (non-unital!) subalgebra of End V. The Lie algebra corresponding to $V \otimes V_*$ via formula (1.2) is nothing but $\mathfrak{gl}(\infty)$, and in fact $V \otimes V_*$ is identified with the algebra of matrices (a_{ij}) that are "infinite down and to the right" (i.e., $i, j \in \mathbb{Z}_{>0}$) and have only finitely many nonzero entries. More precisely, a nonzero matrix of the form (a_{ij}) with $a_{ij} = 0$ for $i \neq i_0$, $j \neq j_0$, is nothing but a generator of the subspace $V^{\varepsilon_{i_0}} \otimes V_*^{-\varepsilon_{j_0}}$ in $V \otimes V_*$. The subalgebra of traceless matrices in $V \otimes V_*$ is then identified with our original Lie algebra $\mathfrak{g}$. In this realization of $\mathfrak{g}$, the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is identified with the algebra of traceless diagonal matrices in $V \otimes V_*$.

Let us now turn our attention to the Borel subalgebras of $\mathfrak{g} = \mathfrak{sl}(\infty)$. The fixed subalgebra $\mathfrak{h}$ is abelian, hence solvable, and is therefore contained (for instance, by Zorn's Lemma) in some maximal locally solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The subalgebra $\mathfrak{b}$ is of course not unique. (Think of the finite-dimensional case!) However, as we shall see in the next chapter, not every maximal locally solvable subalgebra of $\mathfrak{g}$ contains an abelian subalgebra with respect to which $\mathfrak{g}$ admits a root decomposition (3.10). In this chapter we will describe explicitly all Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ that contain the fixed subalgebra $\mathfrak{h}$. By definition, these are *splitting* Borel subalgebrasof $\mathfrak{g}$, where more generally, a splitting Borel subalgebra is defined as a Borel subalgebra containing the image of $\mathfrak{h}$ under some (arbitrary) automorphism of $\mathfrak{g}$.

The above identification of $\mathfrak{g}$ with an explicit matrix algebra enables us to assume that $\mathfrak{h}$ is the subalgebra of traceless diagonal matrices. As in Chap. 3, we have $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\}$.

Our main result in the first half of this chapter is the following statement.

There are natural bijections between the set of (splitting) Borel subalgebras $\mathfrak{b}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{h}$ *and the following two sets:*

- *the set of all subsets* $\Delta_\mathfrak{b} \subset \Delta$ *satisfying conditions (B1) and (B2);*
- *the set of all strict linear orders on the set* $\mathbb{Z}_{>0}$.

We start the proof with the following important (standard) proposition.

Proposition 5.1 *If an* $\mathfrak{h}$*-module* M *is a direct sum of* $\mathfrak{h}$*-eigenspaces, that is*

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda \tag{5.2}$$

where $M^\lambda := \{m \in M \mid h \cdot m = \lambda(h)m \quad \forall h \in \mathfrak{h}\}$*, then any* $\mathfrak{h}$*-submodule* $M' \subset M$ *has an induced decomposition* $M' = \bigoplus_{\lambda \in \mathfrak{h}^*} (M')^\lambda$ *where* $(M')^\lambda = M' \cap M^\lambda$.

A module with the decomposition (5.2) is by definition an $\mathfrak{h}$-*weight module*.

Proof To prove Proposition 5.1 is the same as to show that any $\mathfrak{h}$-submodule M' is also an $\mathfrak{h}$-weight module. Consider an arbitrary nonzero vector $m' \in M$ and

decompose it as $\sum_{i=1}^{n} m_i$ for some n, where $0 \neq m_i \in M^{\lambda_i}$ and the linear functionals λ_i are pairwise distinct. Fix an element $h \in \mathfrak{h}$ such that the values $\lambda_i(h)$ are distinct for $i = 1,\dots, n$. (To the reader: prove using linear algebra that such an element h always exists!) By repeatedly applying h to m', one obtains the linear system

$$\begin{aligned} m' &= \sum_{i=1}^{n} m_i, \\ h \cdot m' &= \sum_{i=1}^{n} \lambda_i(h) m_i, \\ &\vdots \\ h^{n-1} \cdot m' &= \sum_{i=1}^{n} (\lambda_i(h))^{n-1} m_i \end{aligned}$$

with unknown vectors m_i. The matrix Λ of the system is

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1(h) & \lambda_2(h) & & \lambda_n(h) \\ \vdots & \vdots & & \vdots \\ (\lambda_1(h))^{n-1} & (\lambda_2(h))^{n-1} & \dots & (\lambda_n(h))^{n-1} \end{pmatrix},$$

and we can write the system as

$$\begin{pmatrix} m' \\ h \cdot m' \\ \vdots \\ h^{n-1} \cdot m' \end{pmatrix} = \Lambda \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}.$$

Since Λ is a Vandermonde matrix, it is invertible (to the reader: please refresh the Vandermonde determinant in your memory!), and we obtain

$$\Lambda^{-1} \begin{pmatrix} m' \\ h \cdot m' \\ \vdots \\ h^{n-1} \cdot m' \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}.$$

This means that each $m_i \in \mathrm{span}\{m', h\cdot m', \ldots, h^{n-1}\cdot m'\}$, i.e., $m_i \in M'$ as $h^k \cdot m' \in M$ for any k. Consequently

$$M' = \bigoplus_{\lambda\in\mathfrak{h}^*} (M')^\lambda,$$

and we have proved Proposition 5.1. □

Now let $\mathfrak{b} \supset \mathfrak{h}$ be a Borel subalgebra. Then as $\mathfrak{b}$ is an $\mathfrak{h}$-submodule of $\mathfrak{g}$, and $\mathfrak{g}$ is an $\mathfrak{h}$-weight module, $\mathfrak{b}$ is an $\mathfrak{h}$-weight module too. Furthermore, as $\dim \mathfrak{g}^\alpha = 1$ for any $\alpha \in \Delta$, we have

$$\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta_\mathfrak{b}} \mathfrak{g}^\alpha)$$

for some subset $\Delta_\mathfrak{b} \subset \Delta$. Since for each $\alpha \in \Delta$, $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ generate a non-solvable subalgebra of $\mathfrak{g}$ (which is isomorphic to $\mathfrak{sl}(2)$), the condition $\alpha \in \Delta_\mathfrak{b}$ implies $-\alpha \notin \Delta_\mathfrak{b}$. In addition, the set $\Delta_\mathfrak{b}$ is clearly additive within Δ (i.e., $\alpha, \beta \in \Delta_\mathfrak{b}$ and $\alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Delta_\mathfrak{b}$) as $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$. Therefore $\Delta_\mathfrak{b}$ satisfies condition (B1) and a weaker version of condition (B2). In order to establish condition (B2), we need to prove that if $-\alpha \notin \Delta_\mathfrak{b}$ for some $\alpha \in \Delta$, then $\alpha \in \Delta_\mathfrak{b}$.

To prove this latest statement, consider all roots $\varepsilon_i - \varepsilon_j$ that belong to $\Delta_\mathfrak{b}$. If we set $i \prec\prec j$ for $\varepsilon_i - \varepsilon_j$, it is clear that we obtain a strict partial order on the set of indices $\mathbb{Z}_{>0}$. Indeed, the additivity of $\Delta_\mathfrak{b}$ implies that the partial order $\prec\prec$ is transitive, and the fact that $-\alpha \notin \Delta_\mathfrak{b}$ for $\alpha \in \Delta_\mathfrak{b}$ implies that $\prec\prec$ is antisymmetric, i.e., that $i \prec\prec j$ and $j \prec\prec i$ are contradictory. A key observation now is that any strict partial order on $\mathbb{Z}_{>0}$ can be extended to a strict linear order on $\mathbb{Z}_{>0}$. Here is a quick hint why this is true: denote by 1^+ all elements which are greater than 1 with respect to $\prec\prec$, and by 1^- all elements which are smaller than 1 with respect to $\prec\prec$. Let i_0 be the smallest element in $\mathbb{Z}_{>0}$ with respect to the usual order which does not belong to 1^+ or 1^-. Then define the extension $\prec$ of $\prec\prec$ to preserve the usual order between 1 and i_0, i.e., $1 \prec i_0$. One checks that this is obviously compatible with $\prec\prec$, and continues by induction. This defines $\prec$ on all pairs $(1, j)$ for $j \in \mathbb{Z}_{>0}$. One finishes the argument also by induction, defining $\prec$ on all pairs $(2, j)$, etc.

We now have a well-defined strict linear order $\prec$ on $\mathbb{Z}_{>0}$ extending $\prec\prec$. Consider the subalgebra $\mathfrak{h} \oplus (\bigoplus_{i\prec j} \mathfrak{g}^{\varepsilon_i-\varepsilon_j})$ of $\mathfrak{g}$. Clearly this subalgebra contains $\mathfrak{b}$ and is locally solvable. (Please prove this latter claim!) Therefore it coincides with $\mathfrak{b}$ as, by definition, $\mathfrak{b}$ is a maximal locally solvable subalgebra. The fact that $\Delta_\mathfrak{b}$ satisfies condition (B2) follows immediately. Indeed, if $\alpha = \varepsilon_i - \varepsilon_j \in \Delta$, then (as the order $\prec$ is linear) we have either $i \prec j$ or $j \prec i$, i.e., either $\alpha \in \Delta_\mathfrak{b}$ or $-\alpha \in \Delta_\mathfrak{b}$.

In this way we have assigned to each Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{b} \supset \mathfrak{h}$ a subset $\Delta_\mathfrak{b}$ of Δ satisfying conditions (B1) and (B2), as well as a strict linear order on $\mathbb{Z}_{>0}$ from which one can read off the set $\Delta_\mathfrak{b}$.

Note now that the correspondence between strict linear orders on $\mathbb{Z}_{>0}$ and subsets $\Delta_\mathfrak{b} \subset \Delta$ satisfying conditions (B1) and (B2) is a bijection. Indeed, given an order $\prec$ on $\mathbb{Z}_{>0}$, one puts $\Delta_\mathfrak{b} := \{\varepsilon_i - \varepsilon_j \mid i \prec j \in \mathbb{Z}_{>0}\}$, and conversely given $\Delta_\mathfrak{b}$, one puts $i \prec j$ if and only if $\varepsilon_i - \varepsilon_j \in \Delta_\mathfrak{b}$. The conditions (B1) and (B2) correspond precisely to the fact that $\prec$ is a strict linear order on $\mathbb{Z}_{>0}$.

To complete the proof, we need to see that the correspondence between (splitting) Borel subalgebras $\mathfrak{b} \supset \mathfrak{h}$ and subsets $\Delta_\mathfrak{b} \subset \Delta$ satisfying conditions (B1) and (B2) is a bijection. Let $\Delta_\mathfrak{b}$ be a subset of Δ satisfying the conditions (B1) and (B2). Then

$$\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_\mathfrak{b}} \mathfrak{g}^\alpha \right)$$

is a subalgebra of $\mathfrak{g}$ (as $\Delta_\mathfrak{b}$ is additive within Δ, and since $\mathfrak{b}$ is locally solvable as $\alpha \in \Delta_\mathfrak{b}$ implies $-\alpha \notin \Delta_\mathfrak{b}$). Finally, $\mathfrak{b}$ is maximal solvable. Indeed any subalgebra $\mathfrak{l} \subset \mathfrak{g}$ with $\mathfrak{l} \supset \mathfrak{b}$ is of the form $\mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_\mathfrak{l}} \mathfrak{g}^\alpha)$ for some subset $\Delta_\mathfrak{l} \subset \Delta$ with $\Delta_\mathfrak{l} \supset \Delta_\mathfrak{b}$. This follows simply from the fact that $\mathfrak{l}$ is a weight $\mathfrak{h}$-module. Condition (B2) ensures now that any subset $\Delta_\mathfrak{l} \subset \Delta$ in which $\Delta_\mathfrak{b}$ is properly contained has an element α such that $-\alpha \in \Delta_\mathfrak{l}$. Hence $\mathfrak{l}$ cannot be solvable if $\mathfrak{b}$ is a proper subalgebra of $\mathfrak{l}$. Thus $\mathfrak{b}$ is maximal solvable, and the proof of the following theorem is finished.

Theorem 5.2 *The correspondence*

$$\mathfrak{b} \leftrightarrow \Delta_\mathfrak{b}$$

is a bijection between the set of splitting Borel subalgebras of $\mathfrak{sl}(\infty)$ *containing* $\mathfrak{h}$ *and the set of subsets of* Δ *satisfying conditions (B1) and (B2).*

Here are some examples of Borel subalgebras of $\mathfrak{sl}(\infty)$.

Example 5.3 Consider the following strict linear order $\prec$ on $\mathbb{Z}_{>0}$;

$$2 \prec 3 \prec 4 \prec \ldots \prec 1. \tag{5.3}$$

The corresponding Borel subalgebra $\mathfrak{b}$ has roots

$$\Delta_\mathfrak{b} = \{\varepsilon_i - \varepsilon_1, \varepsilon_i - \varepsilon_j \mid 2 \leq i \prec j\}.$$

Note that the simple roots of $\mathfrak{b}$ do not span $\Delta_\mathfrak{b}$: indeed, for a splitting Borel subalgebra $\mathfrak{b}' \subset \mathfrak{sl}(\infty)$, one can define a *simple root* as an indecomposable vector in $\Delta_\mathfrak{b}$, i.e., as a root $\alpha \in \Delta_{\mathfrak{b}'}$ that cannot be obtained as a linear combination of other roots in $\Delta_{\mathfrak{b}'}$ with coefficients from $\mathbb{Z}_{\geq 0}$. Then clearly the simple roots of $\mathfrak{b}$ are $\varepsilon_i - \varepsilon_{i+1}$ for $i \geq 2$, and they do not span $\Delta_\mathfrak{b}$ over $\mathbb{Z}_{\geq 0}$. (Check!) This shows in particular that splitting Borel subalgebras of $\mathfrak{sl}(\infty)$ cannot be described by

extending the usual Dynkin diagrams of $\mathfrak{sl}(n)$ to infinity in a naive way to the left, to the right or in both directions. ◯

The following important example will be used in Chap. 7.

Example 5.4 Consider the following strict linear order $\prec$ on $\mathbb{Z}_{>0}$

$$1 \prec 3 \prec 5 \prec \cdots \prec 6 \prec 4 \prec 2. \tag{5.4}$$

The corresponding Borel subalgebra $\mathfrak{b}$ has roots

$$\Delta_{\mathfrak{b}} = \{\varepsilon_{2i+1} - \varepsilon_{2j+1} \mid i < j\} \bigcup \{\varepsilon_{2i+1} - \varepsilon_{2j}\} \bigcup \{\varepsilon_{2i} - \varepsilon_{2j} \mid j < i\}.$$

This Borel subalgebra is an example of a perfect Borel subalgebra of $\mathfrak{sl}(\infty)$ as defined in [PS9]. It has the property that both the natural representation V and conatural representation V_* are highest weight modules with respect to this Borel subalgebra. Up to an arbitrary bijection of the set $\mathbb{Z}_{>0}$ with itself, these are the only perfect Borel subalgebras. ◯

Here is another example.

Example 5.5 Let $\varphi : \mathbb{Z}_{>0} \to \mathbb{Q}$ be a bijection, and let $\prec$ stand for the strict linear order of $\mathbb{Z}_{>0}$ induced via φ by the usual order on $\mathbb{Q}$. Let $\mathfrak{b} \subset \mathfrak{g}$ be the splitting Borel subalgebra with $\Delta_{\mathfrak{b}} = \{\varepsilon_i - \varepsilon_j \mid i \prec j,\ i, j \in \mathbb{Z}_{>0}\}$. Then the set of simple roots of $\mathfrak{b}$ is empty! ◯

We say that $\mathfrak{b}$ is a *Dynkin Borel subalgebra* of $\mathfrak{g}$ if it is generated by $\mathfrak{h}$ and the simple root spaces, or equivalently, if the simple roots of $\mathfrak{b}$ span $\Delta_{\mathfrak{b}}$.

Example 5.6 Each of the following 3 orderings

$$1 \prec 2 \prec 3 \prec 4 \prec \cdots$$
$$\cdots \prec 4 \prec 3 \prec 2 \prec 1$$
$$\cdots \prec 6 \prec 4 \prec 2 \prec 1 \prec 3 \prec 5 \prec \cdots$$

corresponds to a Dynkin Borel subalgebra of $\mathfrak{sl}(\infty)$. Moreover, up to an arbitrary bijection of the set $\mathbb{Z}_{>0}$ with itself, these are all the Dynkin Borel subalgebras of $\mathfrak{sl}(\infty)$. ◯

5.1.2 *Splitting Borel Subalgebras of* $\mathfrak{o}(\infty)$ *and* $\mathfrak{sp}(\infty)$

It is time now to consider the other two classical locally simple Lie algebras $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. Let $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{g} = \mathfrak{sp}(\infty)$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed subalgebra with $\mathfrak{h} = \mathfrak{g}_{\mathfrak{h}}^0$ for which the generalized root decomposition (3.10) holds. As we know

from Chap. 3, the decomposition (3.10) is necessarily a root decomposition, $\mathfrak{h}$ is abelian, and for the root system Δ, there are the following three possibilities:

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i \neq j \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = \mathfrak{sp}(\infty), \tag{5.5}$$

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid i \neq j \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty), \tag{5.6}$$

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty). \tag{5.7}$$

The corresponding root-isomorphism classes of root-reductive Lie algebras are C_∞, B_∞ and D_∞.

We claim that Theorem 5.2 also holds for C_∞, B_∞ and D_∞ with appropriate modifications. Namely, the combinatorial description of all sets $\Delta_\mathfrak{b}$ through linear orders has to be modified as follows. In all three cases, one considers $\mathbb{Z}$ instead of $\mathbb{Z}_{>0}$. More precisely for B_∞ and C_∞, all splitting Borel subalgebras containing a fixed Cartan subalgebra $\mathfrak{h}$ are in 1-1 correspondence with strict linear orders on $\mathbb{Z}$ with the additional property $a \prec b \Leftrightarrow -b \prec -a$. Given such an order $\prec$ on $\mathbb{Z}$, the corresponding subset $\Delta_\mathfrak{b} \subset \Delta$ for B_∞ is

$$\{\varepsilon_i - \varepsilon_j, \varepsilon_k + \varepsilon_\ell, -\varepsilon_m - \varepsilon_n, \varepsilon_r, -\varepsilon_s \mid i, j, k, \ell, m, n, r, s \in \mathbb{Z}_{>0}, \\ i \prec j, k + \ell \prec 0, 0 \prec m + n, r \prec 0, 0 \prec s\},$$

and for C_∞ is

$$\{\varepsilon_i - \varepsilon_j, \varepsilon_k + \varepsilon_\ell, -\varepsilon_m - \varepsilon_n, 2\varepsilon_r, -2\varepsilon_s \mid i, j, k, \ell, m, n, r, s \in \mathbb{Z}_{>0}, \\ i \prec j, k + \ell \prec 0, 0 \prec m + n, r \prec 0, 0 \prec s\}.$$

For D_∞ one considers only strict linear orders on $\mathbb{Z}$ satisfying $a \prec b \Leftrightarrow -b \prec -a$ and the additional condition that if the element 0 has an immediate successor, then this element belongs to $\mathbb{Z}_{>0}$. To an order $\prec$ on $\mathbb{Z}$, we assign the subset

$$\Delta_\mathfrak{b} := \{\varepsilon_i - \varepsilon_j, \varepsilon_k + \varepsilon_\ell, -\varepsilon_m - \varepsilon_n \mid i, j, k, \ell, m, n \in \mathbb{Z}_{>0}, i \prec j, k + \ell \prec 0 \prec m + n\}.$$

of Δ. It would be a good exercise for the reader to prove the more general version of Theorem 5.2, based on the above correspondences.

5.2 Generalized Flags

In the second half of this chapter, we give another description of all splitting Borel subalgebras of the classical locally simple Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ as stabilizers of maximal generalized flags. But first, we consider briefly the case of $\mathfrak{sl}(n)$.

5.2.1 Finite-Dimensional Flags

If $\mathfrak{b} \subset \mathfrak{sl}(n)$ is a Borel subalgebra and V is the natural representation of $\mathfrak{sl}(n)$, then we can consider V as a $\mathfrak{b}$-module. As $\mathfrak{b}$ is solvable, by Lie's Theorem V has a one-dimensional irreducible submodule $V_1 \subset V$. Furthermore, the quotient V/V_1 also has an irreducible one-dimensional submodule. Denote the two-dimensional preimage of this submodule in V by V_2. Then $V_1 \subset V_2 \subset V$, and by continuing the process we obtain a $\mathfrak{b}$-stable maximal *flag* of subspaces in V:

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V. \tag{5.8}$$

It is a classical fact that the flag (5.8) determines $\mathfrak{b}$. To see this, choose a basis e_1, $e_2, \ldots, e_n$ in V and consider the flag

$$\operatorname{span}\{e_1\} \subset \operatorname{span}\{e_1, e_2\} \subset \cdots \subset \operatorname{span}\{e_1, \ldots, e_{n-1}\} \subset V. \tag{5.9}$$

Let $\sigma : V \to V$ be an automorphism of V such that $\sigma(V_i) = \operatorname{span}\{e_1, \ldots, e_i\}$ for $1 \leq i \leq n-1$. The isomorphism of vector spaces

$$\sigma \otimes \sigma^* : V \otimes V^* \to V \otimes V^*$$

induces an automorphism of $\mathfrak{sl}(n)$ (considered as a Lie subalgebra of $\operatorname{End} V = V \otimes V^*$) and the image $\mathfrak{b}' \subset \mathfrak{sl}(n)$ of $\mathfrak{b}$ under this automorphism is the subalgebra of traceless upper triangular matrices in $V \otimes V^*$. A straightforward computation shows that $\mathfrak{b}'$ equals the stabilizer of the flag (5.9) (please carry out this computation, i.e., show that each subspace in the flag is a $\mathfrak{b}'$-submodule and that $\mathfrak{b}'$ is the unique maximal subalgebra of $\mathfrak{g}$ for which this property holds). It follows then that $\mathfrak{b}$ is the stabilizer of the flag (5.8).

The above discussion leads also to the notion of compatibility between a basis of V and a flag of subspaces in V. We say that a basis $v_1, \ldots, v_n$ of V is *compatible* with a flag

$$V_1' \subset V_2' \subset \cdots \subset V_s' \subset V$$

if for every i, $1 \leq i \leq s$, there are $j_1, \ldots, j_r$ such that $V_i' = \operatorname{span}\{v_{j_1}, \ldots, v_{j_r}\}$, or in other words if every subspace V_i' is a *coordinate space* for the basis v_1, $\ldots$, v_n. Note that a fixed basis has only finitely many coordinate spaces and can thus be compatible with only finitely many flags; however, a fixed flag can be compatible with infinitely many bases. This is directly related to the fact that a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}(n)$ is contained in finitely many (precisely in $n!$) Borel subalgebras of $\mathfrak{g}$, while a Borel subalgebra contains infinitely many Cartan subalgebras. Indeed, fixing a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{sl}(n)$ is equivalent to decomposing V as a direct sum of one-dimensional subspaces (prove this!), and

every Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ is the stabilizer of a flag of maximal length in V compatible with any basis that induces the decomposition of V determined by $\mathfrak{h}$.

The decomposition of Borel subalgebras as stabilizers of maximal flags in the natural representation plays a crucial role in geometric representation theory. This theory relates the geometry of flag varieties with the representation theory of $\mathfrak{g}$. In Sect. 10.1, we will discuss the basics of the geometric representation theory of $\mathfrak{sl}(n)$, and in particular, the Bott–Borel–Weil Theorem for $\mathfrak{sl}(n)$.

5.2.2 Generalized Flags

We are now ready to pursue the "flag approach" to the Borel subalgebras of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. Let us first look at $\mathfrak{g} = \mathfrak{sl}(\infty)$. By a *flag* in the natural representation V of $\mathfrak{sl}(\infty)$, we mean a chain of distinct subspaces of V parameterized by an ordered subset of $\mathbb{Z}$ with the conditions that the intersection of all spaces in the chain equals zero and the union of all spaces in the chain equals V. Up to isomorphism of ordered sets, this yields the following four possibilities:

$$\begin{aligned} \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots, &\qquad \text{two-sided flag,} \\ \{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots, &\qquad \text{infinite ascending flag,} \\ \cdots \subset V_{-2} \subset V_{-1} \subset V_0 = V, &\qquad \text{infinite descending flag,} \\ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V, &\qquad \text{finite flag.} \end{aligned}$$

One drastic difference from the finite-dimensional case is that in the infinite-dimensional case flags are no longer sufficient to describe all splitting Borel subalgebras of $\mathfrak{g}$. Indeed, the following is an example of a (splitting) Borel subalgebra for $\mathfrak{sl}(\infty)$ that is not the stabilizer of any flag in the natural representation of $\mathfrak{sl}(\infty)$.

Example 5.7 Consider the realization of $\mathfrak{sl}(\infty)$ as matrices that are "infinite down and to the right" and have finitely many nonzero entries. Fix $\mathfrak{h}$ to be the diagonal subalgebra, and let $\mathfrak{b} \supset \mathfrak{h}$ be the splitting Borel subalgebra corresponding to the linear order (5.3). It is easy to compute all $\mathfrak{b}$-invariant subspaces of the natural $\mathfrak{sl}(\infty)$-module V. First of all, V is an $\mathfrak{h}$-weight module with one-dimensional weight spaces V^{ε_j}, and any $\mathfrak{h}$-stable subspace is of the form $\operatorname{span}\{V^{\varepsilon_j}\}$ for some subset of indices j. If $V^{\varepsilon_1} \subset \tilde{V}$, where $\tilde{V}$ is now a $\mathfrak{b}$-stable subspace, then $\tilde{V} = \mathfrak{b} \cdot V^{\varepsilon_1} = V$ since $\varepsilon_i - \varepsilon_1 \in \Delta_{\mathfrak{b}}$ for any $i \geq 2$. Therefore no proper $\mathfrak{b}$-stable subspace of V contains V^{ε_1}.

On the other hand, for any $i \geq 2$, $V_i' = \operatorname{span}\{V^{\varepsilon_2}, \ldots, V^{\varepsilon_i}\}$ is a $\mathfrak{b}$-stable subspace. This yields a maximal chain of $\mathfrak{b}$-stable subspaces

$$0 \subset V_2' \subset V_3' \subset \cdots \subset V' \subset V, \tag{5.10}$$

where $V' := \bigcup_{i \geq 2} V'_i$. Moreover, any nontrivial $\mathfrak{b}$-stable subspace $\tilde{V}$ is one of the spaces in (5.10). Indeed, $\tilde{V} \cap V' \neq 0$, as V' clearly has no $\mathfrak{b}$-stable (one-dimensional) complement. If there is a maximal k for which $V'_k \subset \tilde{V}$, then $\tilde{V} = V'_k$ (check it!) and if there is no such k, then $\tilde{V}$ equals V' or V. (Check this too!) However, the index set of the chain (5.10) is clearly not isomorphic as an ordered set to a subset of $\mathbb{Z}$. Hence this chain is not a flag. The reader should further check that $\mathfrak{b}$ is in fact the stabilizer of (5.10), and that $\mathfrak{b}$ is not the stabilizer of any proper subchain of (5.10). Therefore $\mathfrak{b}$ is not the stabilizer of a flag of subspaces in V. ◯

We now introduce a sufficiently large class of chains of subspaces in V for which all splitting Borel subalgebras of $\mathfrak{sl}(\infty)$ are stabilizers of such chains. First, a *chain* $C = \{C_\kappa\}$ of subspaces in V is a set of pairwise distinct subspaces $\{C_\kappa\}$ with the property that for any $\mu \neq \nu$, $C_\mu \subset C_\nu$ or $C_\nu \subset C_\mu$. The set of indices of a chain is naturally totally ordered (the order comes from the inclusions of subspaces). For instance, the index set of the chain (5.10) is isomorphic to $\mathbb{Z}_{>0}$ as an ordered set with the order (5.3).

A chain $C = \{C_\kappa\}$ is a *generalized flag* if it satisfies the following two properties:

(1) every index κ has an immediate predecessor or an immediate successor in the index set of C;
(2) for every $v \in V$, there is a pair $C_{\kappa'}, C_{\kappa''} \in C$ such that κ' is the immediate predecessor of κ'' and $v \in C_{\kappa''} \setminus C_{\kappa'}$.

In what follows, we reserve the notation κ', κ'' for pairs of indices such that κ' is the immediate predecessor of κ''.

Clearly, any flag of subspaces of V is a generalized flag. The converse is not true as for instance the chain (5.10) is a generalized flag that is not a flag. We call a generalized flag *maximal* if as a set, it is not properly contained in any chain C of subspaces in V that itself is a generalized flag. It is straightforward to show that a generalized flag $\mathcal{F}$ is maximal if $\dim \mathcal{F}_{\kappa''}/\mathcal{F}_{\kappa'} = 1$ for all pairs κ', κ''.

A basis $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ of V is *compatible* with a generalized flag $\mathcal{F} = \{\mathcal{F}_\kappa\}$ if there is a strict linear order $\prec$ on $\mathbb{Z}_{>0}$, such that for any space of the form $\mathcal{F}_{\kappa'}$ there exists $j \in \mathbb{Z}_{>0}$ such that $\mathcal{F}_{\kappa'} = \operatorname{span}\{e_i\}_{i \prec j}$, and for any space of the form $\mathcal{F}_{\kappa''}$ there exists $k \in \mathbb{Z}_{>0}$ for which $\mathcal{F}_{\kappa''} = \operatorname{span}\{e_k, e_i\}_{i \prec k}$. In particular, all spaces $\mathcal{F}_\kappa$ (and in particular, all spaces of the form $\mathcal{F}_{\kappa''}$) are coordinate spaces with respect to the basis $\{e_i\}_{i \in \mathbb{Z}_{>0}}$. To the reader: show that for a basis, the condition that all spaces $\mathcal{F}_\kappa$ be coordinate spaces is equivalent to the basis being compatible. Moreover, if the generalized flag $\mathcal{F}$ is maximal, then the order $\prec$ is uniquely determined.

Note that our fixed diagonal subalgebra $\mathfrak{h} \subset \mathfrak{g}$ determines a decomposition $V = \bigoplus_i V^{\varepsilon_i}$, with $\dim V^{\varepsilon_i} = 1$ for all i. Fix a basis $e_1, e_2, \ldots$ of V such that $e_i \in V^{\varepsilon_i}$ for each i. We are now ready to state the description of splitting Borel subalgebras of $\mathfrak{g}$ in terms of generalized flags.

Theorem 5.8 *The splitting Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ are in bijection with all maximal generalized flags $\mathcal{F} = \{\mathcal{F}_\kappa\}$ in V compatible with the fixed basis $\{e_i\}$ of V.*

The proof of Theorem 5.8 is given in [DP3]. Here we just construct the bijection in question. In fact, all we need is the following simple construction. Let $\prec$ be a strict linear order on $\mathbb{Z}_{>0}$. For each $j \in \mathbb{Z}_{>0}$, define a pair of subspaces

$$F_{j'} := \operatorname{span}\{e_i \mid i \prec j\},$$
$$F_{j''} := \operatorname{span}\{F_{j'}, e_j\},$$

and set $\mathcal{F}_\prec := \bigcup_j \{F_{j'}, F_{j''}\}$. To prove the theorem, one has to check that the following chain of correspondences is a chain of bijections whose composition equals the identity:

$$\mathfrak{b} \longmapsto \Delta_\mathfrak{b} \longmapsto \prec \longmapsto \mathcal{F}_\prec \longmapsto \mathfrak{b}.$$

The last map assigning $\mathfrak{b}$ back to $\mathcal{F}_\prec$ is nothing but "taking stabilizer", i.e., we claim that $\mathfrak{b}$ is reconstructed as the stabilizer in $\mathfrak{g}$ of the chain of subspaces $\mathcal{F}_\prec$ in V. We challenge the reader to fill in the details.

5.2.3 Maximal Chains and Parabolic Subalgebras of $\mathfrak{sl}(\infty)$

We have described all three approaches to the splitting Borel subalgebras of $\mathfrak{sl}(\infty)$, and now we would like to make a few comments.

Our first comment concerns the idea of replacing maximal generalized flags with chains of subspaces that are maximal in the class of all chains. Although maximal chains of subspaces in V can be uncountable and may have quite a complicated structure, it is in fact true that any splitting Borel subalgebra $\mathfrak{b}$ is the stabilizer of a unique maximal chain ℓ of subspaces of V. The proof of this result can be found in [DP1]. Sometimes a maximal generalized flag in V is also a maximal chain, but sometimes it is not. For instance, the chain (5.10) is a maximal generalized flag and it is also a maximal chain. However, in Example 5.5 where the order $\prec$ is the pullback on $\mathbb{Z}_{>0}$ of the usual order on $\mathbb{Q}$ via a bijection $\varphi : \mathbb{Z}_{>0} \to \mathbb{Q}$, the corresponding generalized flag $\mathcal{F}_\prec$ is not a maximal chain. Indeed, the spaces in $\mathcal{F}_\prec$ are parameterized by two copies of $\mathbb{Q}$, since these spaces are $\mathcal{F}_{q'} = \{e_i \mid i \prec q\}$ and $\mathcal{F}_{q''} = \{e_q, e_i \mid i \prec q\}$ for $q \in \mathbb{Z}_{>0}$. On the other hand, the unique maximal chain $C_\prec$ containing $\mathcal{F}_\prec$ is the union of $\mathcal{F}_\prec$ and all the spaces $C_r := \operatorname{span}\{C_{\varphi^{-1}(p)} \mid p < r\}$ for $r \in \mathbb{R}\backslash\mathbb{Q}$. (Please prove this statement!) In other words, the index set of the chain $C_\prec$ is "$\mathbb{R}$ with all rational points doubled". This is obviously a variation of the construction of $\mathbb{R}$ via Dedekind cuts, the only difference being that the usual ambiguity, as to whether a rational point should belong to the left or to the right part of the cut, is eliminated. A more detailed discussion of this example and a proof

of the fact that in general there is a bijection between maximal generalized flags of subspaces in V and maximal chains of subspaces in V is contained in [DP3].

Our second comment concerns parabolic subalgebras of $\mathfrak{g} = \mathfrak{sl}(\infty)$. Indeed, what are parabolic subalgebras of $\mathfrak{sl}(\infty)$? We will answer this question in the next chapter, here we just discuss the important class of splitting parabolic subalgebras. By definition, a *splitting parabolic subalgebra* is a subalgebra of $\mathfrak{g}$ that contains a splitting Borel subalgebra. A result of [DP1] implies that all splitting parabolic subalgebras of $\mathfrak{g}$ containing the fixed subalgebra $\mathfrak{h}$ are in 1-1 correspondence with all generalized flags of subspaces in V that are compatible with the basis $\{e_i\}$ determined by $\mathfrak{h}$. In addition, these parabolic subalgebras are in 1-1 correspondence with strict partial orders on $\mathbb{Z}_{>0}$ with the property that for each $i \in \mathbb{Z}_{>0}$, the relation "j is not comparable with k" is an equivalence relation. This result is an ultimate justification for the notion of a generalized flag, as maximal chains are less suited for describing splitting parabolic subalgebras.

5.2.4 *The Case of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$*

We will complete this chapter with a quick discussion on the generalized flag approach for the Lie algebras $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. In the finite-dimensional case, i.e., for $\mathfrak{o}(n)$ and $\mathfrak{sp}(2k)$, in the presence of a bilinear form $(\cdot\,,\,\cdot)$ on the natural representation V, the notion of a flag is replaced by the notion of an isotropic flag. We give the corresponding definitions directly in the infinite-dimensional case. The reader unfamiliar with this approach in the finite-dimensional case should read the rest of the chapter and then work out in detail the cases of B_2 and C_3. After that, the general statements will immediately start to make sense.

Let V be the natural representation of B_∞, D_∞ or C_∞, which by definition is just the union of natural representations of B_i, D_i or C_i (we assume that an exhaustion of $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ is fixed,

$$\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_i \subset \mathfrak{s}_{i+1} \subset \cdots, \qquad \mathfrak{g} = \bigcup_i \mathfrak{s}_i$$

where all $\mathfrak{s}_i$ are of the same type B, D, or C, and all inclusions $\mathfrak{s}_i \subset \mathfrak{s}_{i+1}$ are root injections). The space V is equipped with a nondegenerate form $(\cdot\,,\,\cdot)$ which is symmetric for B and D, and skew-symmetric for C. If $U \subset V$ is a subspace, then $U^\perp$ denotes its orthogonal space with respect to $(\cdot\,,\,\cdot)$. A generalized flag $\mathcal{F} = \{\mathcal{F}_{\kappa'}, \mathcal{F}_{\kappa''}\}$ in V is *isotropic* if the map $F \to F^\perp$, for $F \in \mathcal{F}$, is a well-defined involution on $\mathcal{F}$.

To be able to state the main result, we also need to introduce certain special bases of V. We say that a basis of V is *of type B* if it is of the form $\{e_n, e_0, e^n\}$ for $n \in \mathbb{Z}_{>0}$, $(e_0, e_0) = 1$, $(e_i, e^i) = 1$ and for $i, j \in \mathbb{Z}_{>0}$, $i \neq j$,

$$(e_i, e^j) = (e_i, e_j) = (e_i, e_i) = (e^i, e^j) = (e^j, e^j) = (e_0, e_i) = (e_0, e^j) = 0,$$

where the form $(\cdot\,,\,\cdot)$ is symmetric. A basis of the form $\{e_n, e^n\}$ for $n \in \mathbb{Z}_{>0}$ with $(e_i, e^i) = 1$ and for $i, j \in \mathbb{Z}_{>0}$, $i \neq j$,

$$(e_i, e^j) = (e_i, e_j) = (e_i, e_i) = (e^i, e^j) = (e^j, e^j) = 0$$

is *of type C or D* if the form $(\cdot\,,\,\cdot)$ is respectively skew-symmetric or symmetric. It is true that V always admits a respective basis of type B, C or D. The proof is an infinite-dimensional version of the usual orthogonalization procedure and could be an excellent exercise for the reader. One can also look it up in [DP3].

The main result about splitting parabolic subalgebras of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ is the following. Let $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. For any generalized $\mathfrak{h}$-root decomposition (3.10), there exists a (non-unique) basis of V which is of type B, C or D, such that the splitting parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ with the property $\mathfrak{p} \supset \mathfrak{h}$ (i.e., all subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ that contain a maximal locally solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ with $\mathfrak{b} \supset \mathfrak{h}$) are in 1-1 correspondence with all isotopic generalized flags in V compatible with this basis. Given such a generalized flag, the corresponding splitting parabolic subalgebra is just its stabilizer. Splitting Borel subalgebras are in 1-1 correspondence with isotropic maximal generalized flags compatible with the basis. The proofs can be found in [DP4].

Chapter 6
General Cartan, Borel and Parabolic Subalgebras of $\mathfrak{gl}(\infty)$ and $\mathfrak{sl}(\infty)$

In the first part of this chapter, we generalize the theory of Cartan subalgebras from $\mathfrak{sl}(n)$ to $\mathfrak{sl}(\infty)$ and $\mathfrak{gl}(\infty)$. In the second part, we discuss the non-splitting Borel and parabolic subalgebras of $\mathfrak{sl}(\infty)$ and $\mathfrak{gl}(\infty)$.

6.1 Cartan Subalgebras of $\mathfrak{sl}(\infty)$ and $\mathfrak{gl}(\infty)$

We have already considered a natural class of subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{sl}(\infty)$ that can play the role of Cartan subalgebras for $\mathfrak{g}$, namely subalgebras $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}}$ and $\mathfrak{g}$ admits a generalized $\mathfrak{h}$-root decomposition (3.10). Theorem 3.3 in Chap. 3 implies that any such subalgebra $\mathfrak{h}$ is simply the union $\bigcup_i \mathfrak{h}_i$ of Cartan subalgebras $\mathfrak{h}_i$ of $\mathfrak{g}_i \simeq \mathfrak{sl}(i)$ for some exhaustion $\mathfrak{g} = \bigcup_i \mathfrak{g}_i$. It is clear that such a union $\bigcup_i \mathfrak{h}_i$ of Cartan subalgebras of $\mathfrak{g}_i$ should be considered as a Cartan subalgebra of $\mathfrak{g}$, however we have not yet given a general definition of a Cartan subalgebra of $\mathfrak{g}$. A key question for this chapter is: what is a Cartan subalgebra of $\mathfrak{sl}(\infty)$, and is it true that any Cartan subalgebra of $\mathfrak{sl}(\infty)$ equals the union of Cartan subalgebras $\mathfrak{h}_i \subset \mathfrak{g}_i$ for a suitable exhaustion $\mathfrak{g} = \bigcup_i \mathfrak{g}_i$ with $\mathfrak{g}_i \simeq \mathfrak{sl}(i)$?

Recall that a Cartan subalgebra of a finite-dimensional reductive Lie algebra is defined as a self-normalizing nilpotent subalgebra (see Chap. 1, Sect. 1.2). It is then a nontrivial theorem that any Cartan subalgebra is actually a maximal toral subalgebra. It is natural to ask whether this theorem also holds for $\mathfrak{sl}(\infty)$, if one relaxes the condition of nilpotence to local nilpotence. In other words, is it true that any self-normalizing locally nilpotent subalgebra of $\mathfrak{sl}(\infty)$ is maximal toral, and conversely, is it true that any maximal toral subalgebra of $\mathfrak{sl}(\infty)$ is self-normalizing and locally nilpotent?

It turns out that the answer to both of these questions is no, and we will provide corresponding counter-examples in this chapter. Let's start with some preliminaries.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_6

6.1.1 Toral Subalgebras

In the rest of this chapter, we denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{gl}(\infty)$, which we identify with the Lie algebra associated with the associative algebra $V \otimes V_*$, where V and V_* are as in Chap. 5, Sect. 5.1.1. Then $\mathfrak{sl}(\infty)$ is the subalgebra of $\mathfrak{g}$ consisting of traceless matrices (a_{ij}) (by definition, $a_{ij} \neq 0$ for only finitely many pairs $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$). Note that the identification $\mathfrak{g} = V \otimes V_*$ is compatible with the $\mathfrak{g}$-module structure, i.e., that the adjoint $\mathfrak{g}$-module is isomorphic to the tensor product of the $\mathfrak{g}$-modules V and V_*. (Check this by a direct computation!)

A *toral subalgebra* of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{t} \subset \mathfrak{g}$, such that for every nonzero element $t \in \mathfrak{t}$, the linear operator

$$[t, \cdot\,] : \mathfrak{g} \to \mathfrak{g}$$

is semisimple, i.e., diagonalizable.

Lemma 6.1 *Any toral subalgebra of* $\mathfrak{g}$ *is abelian.*

Proof Consider two nonzero elements $t_1, t_2 \in \mathfrak{t}$, and let $t_2 = \sum_s t_2^s$ be the decomposition of t_2 as a linear combination of $[t_1, \cdot\,]$-eigenvectors with distinct eigenvalues. Then $[t_1, t_2] = \sum_s \lambda_s t_2^s$, where λ_s is the eigenvalue of t_2^s. Moreover, by the same argument as in the proof of Proposition 5.1, which uses the Vandermonde matrix $t_2^s \in \mathfrak{t}$ for all s. Since $[t_2^s, [t_2^s, t_1]] = [t_2^s, -\lambda_s t_2^s] = -\lambda_s [t_2^s, t_2^s] = 0$, and since $[t_2^s, \cdot\,]$ is a semisimple operator on $\mathfrak{t}$, we have $[t_2^s, t_1] = 0$ for any s. Hence $[t_2, t_1] = \sum_s [t_2^s, t_1] = 0$. □

Lemma 6.2 *If* $t \in \mathfrak{g}$*, then the linear operator* $[t, \cdot\,] : \mathfrak{g} \to \mathfrak{g}$ *is semisimple if and only if* t *is semisimple as an endomorphism of* V*, or equivalently of* V_**.*

Proof Let $t \in \mathfrak{g}$ be a semisimple endomorphism of V. Then $V = V_t \oplus W_t$ where V_t is the image of t (i.e., the direct sum of t-eigenspaces with nonzero eigenvalues) and W_t is the kernel of t. Clearly, $\dim V_t < \infty$ and $\dim W_t = \infty$. Similarly, $V_* = (V_*)_t \oplus (W_*)_t$, and therefore the tensor product $V \otimes V_*$ equals the direct sum of t-invariant spaces $(V_t \otimes (V_*)_t) \oplus (V_t \otimes (W_*)_t) \oplus (W_t \otimes (V_*)_t) \oplus (W_t \otimes (W_*)_t)$, each of which is clearly a direct sum of t-eigenspaces. Hence $[t, \cdot\,] : V \otimes V_* \to V \otimes V_*$ is semisimple. The case when t is a semisimple endomorphism of V_* is analogous.

Conversely, let $[t, \cdot\,] : \mathfrak{g} \to \mathfrak{g}$ be semisimple. Let $V_t \subset V$ and $(V_*)_t \subset V_*$ be finite-dimensional t-invariant subspaces with respective complements $W_t \subset V$ and $(W_*)_t \subset V_*$ on which t acts by zero. Such spaces exist for any $t \in \mathfrak{g}$, i.e., the semisimplicity assumption on $[t, \cdot\,]$ is not necessary here. In addition, one can always assume that the pairing $V_t \times (V_*)_t \to \mathbb{C}$ is nondegenerate and that $t \in V_t \otimes (V_*)_t$. Since $V_t \otimes (V_*)_t$ is a $[t, \cdot\,]$-invariant subspace of $\mathfrak{g}$, the semisimplicity of $[t, \cdot\,]$ implies that t is semisimple as an element of the finite-dimensional Lie algebra $V_t \otimes (V_*)_t$ that is isomorphic to $\mathfrak{gl}(\dim V_t)$. Then by Bourbaki [B, Ch I, 6, no.3, Theorem 3] (applied to the projection of t to $\mathfrak{sl}(\dim V_t)$, please work out the details!), $[t, \cdot\,]$ is a semisimple operator on any irreducible finite-dimensional

representation of $V_t \otimes (V_*)_t$, and in particular on V_t and $(V_*)_t$. (The reader should attempt to check the latter statement directly for the natural representation of $\mathfrak{sl}(n)$ and its dual, which is all we need here). Hence t is a semisimple endomorphism of both V and V_*. □

The subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, consisting of all diagonal matrices belonging to $\mathfrak{g}$, is toral, (recall decomposition (5.1) and the decomposition $V_* = \bigoplus_i V_*^{-\varepsilon_i}$) and is clearly maximal toral. Furthermore, $\mathfrak{h}$ is self-normalizing, and as it is abelian, $\mathfrak{h}$ is also locally nilpotent. More generally, if $\mathfrak{g} = \bigcup_i \mathfrak{g}_i$ is any exhaustion of $\mathfrak{g}$ as a union of $\mathfrak{gl}(i)$ for $i \to \infty$, and if $\mathfrak{h} = \bigcup_i \mathfrak{h}_i$, each $\mathfrak{h}_i$ being a Cartan subalgebra of $\mathfrak{gl}(i)$, then $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$ that is self-normalizing.

Recall that if $\mathfrak{g}$ were finite dimensional, i.e., $\mathfrak{g} = \mathfrak{gl}(V)$ with $\dim V < \infty$, then every maximal toral subalgebra of $\mathfrak{g}$ would be self-normalizing. Moreover, there would be a 1-1 correspondence between the maximal toral subalgebras of $\mathfrak{g}$ and pairs of dual bases of V and V_* (up to scaling). Indeed, when $\dim \mathfrak{g} < \infty$, a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ determines a decomposition of V and V_* into one-dimensional weight spaces, and respective (nonzero) weight vectors $v_\lambda \in V^\lambda$ and $v_\lambda^* \in V_*^{-\lambda}$ can be chosen to form a pair of dual bases. Such a pair of dual bases is determined up to scaling. Conversely, a pair of dual bases of V and V_* determines a matrix presentation of $\mathfrak{gl}(V)$, such that the set of diagonal matrices is a maximal toral subalgebra $\mathfrak{h}$, and the given dual bases are $\mathfrak{h}$-weight vectors.

Here is an example of a maximal toral subalgebra of $\mathfrak{g}$ that is not self-normalizing, and hence there is no exhaustion of $\mathfrak{g}$ by subalgebras $\mathfrak{g}_n$ isomorphic to $\mathfrak{gl}(n)$ such that $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$ for Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{g}_n$.

Example 6.3 Fix nonzero vectors $e_i \in V^{\varepsilon_i}$ and $e_i^* \in V_*^{-\varepsilon_i}$ with $e_i^*(e_i) = 1$ for $i \in \mathbb{Z}_{>0}$. Then automatically $e_i^*(e_j) = 0$ for $i \neq j$. Consider the following subspaces of V and V_*:

$$V^i = \begin{cases} \operatorname{span}\{e_i + e_1\} & \text{for } i \geq 3,\ i \text{ odd} \\ \operatorname{span}\{e_i\} & \text{for } i \geq 4,\ i \text{ even,} \end{cases}$$
$$V_*^i = \begin{cases} \operatorname{span}\{e_i^*\} & \text{for } i \geq 3,\ i \text{ odd} \\ \operatorname{span}\{e_i^* + e_2^*\} & \text{for } i \geq 4,\ i \text{ even.} \end{cases} \tag{6.1}$$

We claim that $\mathfrak{t} := \bigoplus_{i \geq 3} (V^i \otimes V_*^i)$ is a maximal toral subalgebra of $\mathfrak{g}$. To prove this, consider the exhaustion

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots, \quad \bigcup_i \mathfrak{g}_i = \mathfrak{g},$$

where $\mathfrak{g}_i = (\operatorname{span}\{e_1, \ldots, e_i\}) \otimes (\operatorname{span}\{e_1^*, \ldots, e_i^*\})$. Clearly, the pair of sets $\{e_3+e_1, e_4, e_5+e_1, \ldots, e_i+e_1\}$, $\{e_3^*, e_4^*+e_2^*, e_5^*, \ldots, e_i^*\}$ for i odd, and respectively $\{e_3 + e_1, e_4, e_5 + e_1, \ldots, e_i\}$, $\{e_3^*, e_4^* + e_2^*, e_5^*, \ldots, e_i^* + e_2^*\}$ for i even, can be extended to a respective pair of dual bases of $\operatorname{span}\{e_1, \ldots, e_i\}$ and $\operatorname{span}\{e_1^*, \ldots, e_i^*\}$.

The intersection $\mathfrak{t} \cap \mathfrak{g}_i$ is then a subalgebra of the unique maximal toral subalgebra of $\mathfrak{g}_i$ whose eigenspaces in $\operatorname{span}\{e_1, \ldots, e_i\}$ and $\operatorname{span}\{e_1^*, \ldots, e_i^*\}$ are spanned by precisely one element of the respective bases. In particular, each $\mathfrak{t}_i := \mathfrak{t} \cap \mathfrak{g}_i$ is toral, hence $\mathfrak{t}$ is toral.

To see that $\mathfrak{t}$ is maximal toral, note that any toral subalgebra $\tilde{\mathfrak{t}} \subset \mathfrak{g}$ with $\tilde{\mathfrak{t}} \supset \mathfrak{t}$ intersects $\mathfrak{g}_i$ in a toral subalgebra containing $\mathfrak{t}_i$ and arises from a pair of dual bases as above. Moreover any such pair of dual bases contains vectors of the form $v \in V$ with $e_2^*(v) \neq 0$, and $v_* \in V_*$ with $v_*(e_1) \neq 0$. The intersection $\tilde{\mathfrak{t}}_i := \tilde{\mathfrak{t}} \cap \mathfrak{g}_i$ equals $\mathfrak{t}_i$ if and only if $\operatorname{span}\{e_1\}$ and $\operatorname{span}\{e_2^*\}$ are both eigenspaces of $\tilde{\mathfrak{t}}_i$ of weight zero. (Prove this by a direct computation!) Now if we assume that $\mathfrak{t} \neq \tilde{\mathfrak{t}}$, we obtain an element $\tilde{t} \in \tilde{\mathfrak{t}}$ with $\tilde{t} \cdot e_1 \neq 0$ or $\tilde{t} \cdot e_2^* \neq 0$. Then $\tilde{t}$ does not commute with elements of $\mathfrak{t}$ of the form $(e_j + e_1) \otimes e_j^*$ or $e_j \otimes (e_j^* + e_2^*)$ for $j > i$, which is a contradiction. Therefore $\tilde{\mathfrak{t}} \cap \mathfrak{g}_i = \mathfrak{t}_i$ for all i, i.e., $\tilde{\mathfrak{t}} = \mathfrak{t}$ and $\mathfrak{t}$ is maximal toral.

The maximal toral subalgebra $\mathfrak{t}$ is of course locally nilpotent (being abelian), but it is clearly not self-normalizing as the element $e_1 \otimes e_2^*$ belongs to the centralizer $C(\mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{g}$, but does not belong to $\mathfrak{t}$. It is in fact true that $C(\mathfrak{t})$ equals $\operatorname{span}\{e_1 \otimes e_2^*\} \oplus \mathfrak{t}$, and that $C(\mathfrak{t})$ is a self-normalizing abelian subalgebra of $\mathfrak{g}$. This follows from our general considerations below; nevertheless we urge the reader to verify it by a direct computation. ◯

6.1.2 *Dual Systems*

Maximal toral subalgebras are in general not self-normalizing, as was seen in Example 6.3. However as we shall prove in Theorem 6.9 below, general Cartan subalgebras are nothing but centralizers of arbitrary maximal toral subalgebras. To get a feeling for what this theorem is really saying, we will first study the centralizers of maximal toral subalgebras of $\mathfrak{g} = \mathfrak{gl}(\infty)$. For this purpose, we need the following definition. If $\mathfrak{k}$ is a Lie algebra and M is a $\mathfrak{k}$-module, we call M *locally* $\mathfrak{k}$*-finite* if $\dim(U(\mathfrak{k}) \cdot m) < \infty$ for any $m \in M$, where $U(\mathfrak{k})$ denotes the universal enveloping algebra of $\mathfrak{k}$.

Assume now that $\mathfrak{t} \subset \mathfrak{g}$ is an arbitrary infinite-dimensional toral subalgebra. Then by Lemmas 6.1 and 6.2, $\mathfrak{t}$ is an abelian Lie algebra of semisimple endomorphisms of V, and if $\mathfrak{t}$ were finite dimensional, then all endomorphisms in $\mathfrak{t}$ would be simultaneously diagonalizable. The crucial difference in the case we are considering is that simultaneous diagonalizability does not hold. More precisely, let $\{V^\alpha\}$ be the set of joint eigenspaces in V of all elements of $\mathfrak{t}$ considered as endomorphisms of V. Then each α is a linear functional on $\mathfrak{t}$, and we denote by $\operatorname{supp}_{\mathfrak{t}} V$ the set of all such linear functionals α with $V_\alpha \neq 0$. The sum of the $\mathfrak{t}$-eigenspaces V^α is direct (prove it!) and

$$V' = \bigoplus_{\alpha \in \operatorname{supp}_{\mathfrak{t}} V} V^\alpha$$

is a $\mathfrak{t}$-submodule of V. Similarly, the direct sum

$$V'_* = \bigoplus_{\beta \in \mathrm{supp}_{\mathfrak{t}} V_*} V_*^{\beta}$$

of all $\mathfrak{t}$-eigenspaces of V_* is a $\mathfrak{t}$-submodule of V_*. However when $\mathfrak{t}$ is not finite dimensional, V' and V'_* are in general proper $\mathfrak{t}$-submodules of V and V_*, respectively. Consider the following example.

Example 6.4 If $\mathfrak{t}$ is the maximal toral subalgebra from Example 6.3, then the spaces V^i are precisely the $\mathfrak{t}$-eigenspaces of V (recall that $V^i = V^{\varepsilon_i}$ for i even, $i \geq 4$, and $V^i = V^{\varepsilon_i + \varepsilon_1}$ for i odd, $i \geq 3$) and the direct sum $\bigoplus_{i \geq 3} V^i$ is a $\mathfrak{t}$-submodule of V of codimension 2. (In this case, the direct sum of the $\mathfrak{t}$-eigenspaces of V_* also has codimension 2.) ◯

Lemma 6.5 *For any toral subalgebra $\mathfrak{t} \subset \mathfrak{g}$, the $\mathfrak{t}$-submodules*

$$V' := \bigoplus_{\alpha \in \mathrm{supp}_{\mathfrak{t}} V} V^{\alpha} \subset V \quad \text{and} \quad V'_* := \bigoplus_{\beta \in \mathrm{supp}_{\mathfrak{t}} V_*} V_*^{\beta} \subset V_*$$

are the unique maximal locally $\mathfrak{t}$-finite submodules of V and V_*, respectively.

Proof Suppose there is a nonzero $v \in V$ such that $\dim U(\mathfrak{t}) \cdot v < \infty$. Then $U(\mathfrak{t}) \cdot v$ is a finite-dimensional $\mathfrak{t}$-invariant subspace, and since $\mathfrak{t}$ is an abelian Lie algebra of semisimple endomorphisms of $U(\mathfrak{t}) \cdot v$, we have $U(\mathfrak{t}) \cdot v = \bigoplus_{\alpha} (U(\mathfrak{t}) \cdot v)^{\alpha}$ for some linear functionals α on $\mathfrak{t}$. Then $(U(\mathfrak{t}) \cdot v)^{\alpha} \subset V^{\alpha}$ for each α, and hence $v \in V'$. The same argument applies to V_*. □

We need also the following important general lemma due to I. Dimitrov and G. Zuckerman. Its proof first appeared in [NP].

Lemma 6.6 *Let $\mathfrak{k}$ be a Lie algebra and let the superscript $'$ indicate the unique maximal locally $\mathfrak{k}$-finite submodule of a $\mathfrak{k}$-module. Then for any $\mathfrak{k}$-modules M and N,*

$$(M \otimes N)' = M' \otimes N'.$$

Before proving the lemma, we introduce some notation. Let U and W be vector spaces. To any element $x \in U \otimes W$ we assign a subspace $U_x \subset U$ in the following way. Write x as $\sum_j u_j \otimes w_j$ with linearly independent $w_j \in W$, and set $U_x := \mathrm{span}\{u_j\}$. In a similar way we assign to each $x \in U \otimes W$ a subspace $W_x \subset W$. To check that U_x (and similarly W_x) does not depend on the presentation of x as $\sum_j u_j \otimes w_j$, it suffices to identify U_x with the image of the linear operator $\psi(x) \in \mathrm{Hom}(W^*, U)$, where ψ is the canonical inclusion

$$U \otimes W \hookrightarrow \mathrm{Hom}(W^*, U), \quad \psi(u \otimes w)(\alpha) := \alpha(w)u.$$

This is a straightforward check which we omit.

It is clear that $\dim U_x < \infty$. Note also that for any subspace $Y \subset U$, we have

$$Y \otimes W = \{x \in U \otimes W : U_x \subset Y\},$$

and similarly for any $Z \subset W$,

$$U \otimes Z = \{x \in U \otimes W : W_x \subset Z\}.$$

In particular, $x \in U_x \otimes W_x$.

We are now ready for the main argument.

Proof of Lemma 6.6 Fix a nonzero element $x \in (M \otimes N)'$ and a basis $x_1, \ldots, x_n$ of $U(\mathfrak{k}) \cdot x$ with $x_1 = x$. Set $Y := M_{x_1} + \cdots + M_{x_n}$ and $Z := N_{x_1} + \cdots + N_{x_n}$. Since $x \in M_x \otimes N_x \subset Y \otimes Z$, it suffices to prove that $Y \subset M'$ and $Z \subset N'$. We will show that $Z \subset N'$ (the argument for Y is completely similar), which will follow from the inclusion $U(\mathfrak{k}) \cdot Z \subset Z$. To establish this inclusion, it is enough to verify that $k \cdot N_{x_i} \subset Z$ for any i and any $k \in \mathfrak{k}$.

Fix k and i, and write x_i as $\sum_j m_j \otimes n_j$ with linearly independent $m_j \in M$. Then $k \cdot x_i = \sum_j k \cdot m_j \otimes n_j + \sum_j m_j \otimes k \cdot n_j$. Since $k \cdot x_i \in Y \otimes Z$ and $\sum_j k \cdot m_j \otimes n_j \in M \otimes Z$, we have $\sum_j m_j \otimes k \cdot n_j \in M \otimes Z$. Therefore $N_{\sum_j m_j \otimes k \cdot n_j} \subset Z$, i.e., $k \cdot n_j \in Z$. As the n_j generate N_{x_i}, this implies $k \cdot N_{x_i} \subset Z$. □

Lemmas 6.5 and 6.6 imply that $\mathfrak{g}' := V' \otimes V'_*$ is the unique maximal locally $\mathfrak{t}$-finite submodule of $\mathfrak{g}$, and that $\mathfrak{t} \subset \mathfrak{g}'$ and $C(\mathfrak{t}) \subset \mathfrak{g}'$ (recall that $C(\mathfrak{t})$ denotes the centralizer of $\mathfrak{t}$). The following proposition provides a characterization of maximal toral subalgebras.

Proposition 6.7

(a) A toral subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is maximal if and only if the following two properties hold:

(1) $(\mathrm{supp}_{\mathfrak{t}} V)\backslash\{0\} = -(\mathrm{supp}_{\mathfrak{t}} V_*)\backslash\{0\}$ *and* $\dim V^{\alpha} = \dim V_*^{-\alpha} = 1$ *for any* $\alpha \in (\mathrm{supp}_{\mathfrak{t}} V)\backslash\{0\}$*;*

(2) $V_*^0(V^0) = \{0\}$ *and* $(V^0 \otimes V_*^0) \cap \mathfrak{t} = \{0\}$.

(b) If $\mathfrak{t}$ is a maximal toral subalgebra, then the following three properties also hold:

(3) $\mathfrak{t} = \bigoplus_{\alpha \in (\mathrm{supp}_{\mathfrak{t}} V)\backslash\{0\}} V^{\alpha} \otimes V_*^{-\alpha}$*;*

(4)

$$V_*^0 = \bigcap_{\alpha \in (\mathrm{supp}_{\mathfrak{t}} V)\backslash\{0\}} (V^{\alpha})^{\perp} \text{ and } V^0 = \bigcap_{\beta \in (\mathrm{supp}_{\mathfrak{t}} V_*)\backslash\{0\}} (V_*^{\beta})^{\perp},$$

where for $U \subset V$, $U^{\perp} := \{v_* \in V_* \mid v_*(U) = 0\}$, and for $W \subset V_*$, $W^{\perp} :=$
$\{v \in V \mid W(v) = 0\}$.

(5) $C(\mathfrak{t}) = (V^0 \otimes V_*^0) \oplus \mathfrak{t}$, *and* $C(\mathfrak{t})$ *is a self-normalizing abelian subalgebra of* $\mathfrak{g}$.

Proof Let's first prove *(1)–(5)* under the assumption that $\mathfrak{t}$ is maximal.

(1) We have $V' = \bigoplus_{\alpha \in \mathrm{supp}_{\mathfrak{t}} V} V^\alpha$ and $V'_* = \bigoplus_{\beta \in \mathrm{supp}_{\mathfrak{t}} V_*} V_*^\beta$. It follows from Lemma 6.6 that

$$\mathfrak{g}' = (V \otimes V_*)' = V' \otimes V'_* = \bigoplus_{\alpha \in \mathrm{supp}_{\mathfrak{t}} V,\ \beta \in \mathrm{supp}_{\mathfrak{t}} V'} V^\alpha \otimes V_*^\beta.$$

Since $\mathfrak{t} \subset \mathfrak{g}'$ and $\mathfrak{t}$ is abelian, we obtain

$$\mathfrak{t} \subset \bigoplus_{\alpha \in (\mathrm{supp}_{\mathfrak{t}} V) \cap (-\mathrm{supp}_{\mathfrak{t}} V_*)} V^\alpha \otimes V_*^{-\alpha}.$$

Moreover, $V_*^\beta(V^\alpha) = 0$ for $\beta \neq -\alpha$ (to the reader: justify this equality!) and $\mathfrak{t} \cdot V^\alpha \neq \{0\}$ for $\alpha \in (\mathrm{supp}_{\mathfrak{t}} V) \backslash \{0\}$. So it follows that $V_*^{-\alpha}(V^\alpha) \neq \{0\}$ for all $\alpha \in (\mathrm{supp}_{\mathfrak{t}} V) \backslash \{0\}$. Fix $\alpha \in (\mathrm{supp}_{\mathfrak{t}} V) \backslash \{0\}$, and pick $v \in V^\alpha$ and $\varphi \in V_*^{-\alpha}$ with $\varphi(v) = 1$. Then $v \otimes \varphi$ is a semisimple endomorphism of V that centralizes $\mathfrak{t}$, implying that $v \otimes \varphi \in \mathfrak{t}$ by the maximality of $\mathfrak{t}$. Therefore

$$\{0\} = [v \otimes \varphi, V^\alpha \otimes V_*^{-\alpha}].$$

This implies in particular that

$$\{0\} = [v \otimes \varphi, (\ker \varphi \cap V^\alpha) \otimes V_*^{-\alpha}] = (\ker \varphi \cap V^\alpha) \otimes \varphi.$$

Thus $\ker \varphi \cap V^\alpha = \{0\}$, which yields $\dim V^\alpha = 1$ for $0 \neq \alpha \in (\mathrm{supp}_{\mathfrak{t}} V) \cap (-\mathrm{supp}_{\mathfrak{t}} V_*)$. Similarly, $\dim V_*^{-\alpha} = 1$ for $0 \neq \alpha \in \mathrm{supp}_{\mathfrak{t}} V \cap (-\mathrm{supp}_{\mathfrak{t}} V_*)$. Hence $V^\alpha \otimes V_*^{-\alpha} \subset \mathfrak{t}$.

Finally, as

$$\mathfrak{t} \cdot \left(\bigoplus_{\alpha \in \mathrm{supp}_{\mathfrak{t}} V} V^\alpha \right) = \bigoplus_{\alpha \in \mathrm{supp}_{\mathfrak{t}} V \backslash \{0\}} V^\alpha$$

$$\mathfrak{t} \cdot \left(\bigoplus_{\beta \in \mathrm{supp}_{\mathfrak{t}} V_*} V_*^\beta \right) = \bigoplus_{\beta \in \mathrm{supp}_{\mathfrak{t}} V_* \backslash \{0\}} V_*^\beta,$$

we can also conclude that

$$(\mathrm{supp}_{\mathfrak{t}} V) \backslash \{0\} = -(\mathrm{supp}_{\mathfrak{t}} V_*) \backslash \{0\}.$$

(2) Suppose that there exist $\varphi \in V_*^0$ and $v \in V^0$ with $\varphi(v) = 1$. As above, we see that $v \otimes \varphi \in \mathfrak{t}$, which contradicts $(v \otimes \varphi) \cdot v = \varphi(v)v$ since $v \in V^0$. Therefore $V_*^0(V^0) = \{0\}$, which in turn implies that each endomorphism of V that lies in $V^0 \otimes V_*^0$ is nilpotent. Hence $\mathfrak{t} \cap (V^0 \otimes V_*^0) = \{0\}$.

(3) Since $\mathfrak{t}$ contains all the spaces $V^\alpha \otimes V_*^{-\alpha}$ for $\alpha \neq 0$ and is contained in $\bigoplus_{\alpha \in \operatorname{supp}_{\mathfrak{t}} V} V^\alpha \otimes V_*^{-\alpha}$, we obtain

$$\mathfrak{t} = (\mathfrak{t} \cap (V^0 \otimes V_*^0)) \oplus \bigoplus_{\alpha \in \operatorname{supp}_{\mathfrak{t}} V \setminus \{0\}} V^\alpha \otimes V_*^{-\alpha}.$$

Now *(3)* follows from *(2)*.

(4) In view of *(3)*, the space $\{v \in V \mid \mathfrak{t} \cdot v = \{0\}\}$ coincides with the common annihilator of the spaces $V_*^{-\alpha}$ for $\alpha \neq 0$. Hence,

$$V^0 := \{v \in V \mid t \cdot v = 0\} = \bigcap_{\beta \in (\operatorname{supp}_{\mathfrak{t}} V_*) \setminus \{0\}} (V_*^\beta)^\perp.$$

A similar argument applies to V_*^0.

(5) Note that $C(\mathfrak{t}) \subset \mathfrak{g}'$. Furthermore, Lemma 6.6 implies that $\mathfrak{g}'$ is a weight $\mathfrak{t}$-module, and hence $C(\mathfrak{t})$ is the $\mathfrak{t}$-weight space of weight 0 in $\mathfrak{g}'$. Since $\mathfrak{g}' = V' \otimes V'_*$, clearly this weight space equals $(V^0 \otimes V_*^0) \oplus \mathfrak{t}$. Therefore $C(\mathfrak{t}) = (V^0 \otimes V_*^0) \oplus \mathfrak{t}$, which is abelian since $\mathfrak{t}$ is abelian and $V_*^0(V^\alpha) = V_*^{-\alpha}(V^0) = \{0\}$ for all α. Furthermore, if $N(C(\mathfrak{t}))$ denotes the normalizer of $C(\mathfrak{t})$ in $\mathfrak{g}$, $[N(C(\mathfrak{t})), C(\mathfrak{t})] \subset C(\mathfrak{t})$ implies that $N(C(\mathfrak{t}))$ is a locally $C(\mathfrak{t})$-finite module and hence a locally $\mathfrak{t}$-finite module. Thus $N(C(\mathfrak{t})) \subset \mathfrak{g}'$, and the equality $N(C(\mathfrak{t})) = C(\mathfrak{t})$ follows again from the fact that $C(\mathfrak{t})$ is the $\mathfrak{t}$-weight space of weight 0 in $\mathfrak{g}'$.

Conversely, assume now that *(1)* and *(2)* are satisfied. Then *(3)* is also satisfied as it follows from *(1)* and *(2)*. Therefore $\dim V^\alpha \otimes V_*^{-\alpha} = 1$ for $\alpha \neq 0$ and $V_*^\beta(V^\alpha) = \{0\}$ for $\beta \neq -\alpha$ imply via *(3)* that $\mathfrak{t}$ is abelian and each element $t \in \mathfrak{t}$ is a semisimple endomorphism of V. Hence $\mathfrak{t}$ is a toral subalgebra. Furthermore, the argument in the proof of *(5)* yields

$$C(\mathfrak{t}) = (V^0 \otimes V_*^0) \oplus \mathfrak{t}.$$

Now *(2)* implies that each element in $V^0 \otimes V_*^0$ is nilpotent, hence $\mathfrak{t}$ is a maximal toral subalgebra. □

Proposition 6.7 is a fairly explicit description of maximal toral subalgebras $\mathfrak{t}$ of $\mathfrak{g} = \mathfrak{gl}(\infty)$ and their centralizers. In particular, it claims that any maximal toral subalgebra $\mathfrak{t}$ is determined by the system of one-dimensional subspaces $V^\alpha \subset V$ and $V_*^{-\alpha} \subset V_*$ parameterized by $\operatorname{supp}_{\mathfrak{t}} V \setminus \{0\}$. Such systems of one-dimensional

subspaces can be characterized in a natural self-contained way, and this leads to our final description of maximal toral subalgebras $\mathfrak{t} \subset \mathfrak{g}$.

Consider a pair $(\{V_a\}_{a\in A}, \{(V_*)_a\}_{a\in A})$ of sets of one-dimensional subspaces $V_a \subset V$ and $(V_*)_a \subset V_*$, parameterized by an arbitrary countable set A. The pair $(\{V_a\}_{a\in A}, \{(V_*)_a\}_{a\in A})$ is a *dual system of one-dimensional spaces*, or simply a *dual system*, if $(V_*)_a(V_a) \neq 0$ for any $a \in A$ and $(V_*)_a(V_b) = 0$ for any $a, b \in A$. Dual systems are ordered by inclusion, so the notion of a maximal dual system is well defined.

Proposition 6.8 *The correspondence*

$$\mathfrak{t} \longmapsto \{V^\alpha\}_{\alpha\in(\mathrm{supp}_{\mathfrak{t}}\, V)\setminus\{0\}}, \{V_*^{-\alpha}\}_{\alpha\in(\mathrm{supp}_{\mathfrak{t}}\, V)\setminus\{0\}},$$

$$(\{V_a\}_{a\in A}, \{(V_*)_a\}_{a\in A}) \longmapsto \bigoplus_{a\in A}(V_a \otimes (V_*)_a)$$

yields mutually inverse bijections between the set of maximal toral subalgebras of $\mathfrak{g}$ *and the set of maximal dual systems.*

The proof can be found in [NP]. A key argument is that a dual system is maximal if and only if $V^0 := \bigcap_{a\in A}((V_*)_a)^\perp$ and $V_*^0 := \bigcap_{a\in A}(V_a)^\perp$ are mutually orthogonal, i.e., $V_*^0(V^0) = 0$. We urge the reader to work out the proof on his/her own.

6.1.3 Cartan Subalgebras

We are ready now to state and prove a result about Cartan subalgebras of a very general class of locally finite Lie algebras. Call a Lie algebra $\mathfrak{g}$ *locally reductive* if $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is a finite-dimensional Lie algebra reductive in $\mathfrak{g}_{i+1}$ (i.e., the adjoint action of $\mathfrak{g}_i$ on $\mathfrak{g}_{i+1}$ is semisimple; this implies in particular that $\mathfrak{g}_i$ is reductive). All unions of reductive Lie algebras we have considered so far are locally reductive, in particular any locally simple Lie algebra is automatically locally reductive.

If $\mathfrak{g}$ is locally reductive and $h \in \mathfrak{g}$ is any element of $\mathfrak{g}$, then h has a well-defined Jordan decomposition $h = h_s + h_n$, where h_s is the semisimple component of h and where h_n is the nilpotent component of h. To give a definition, one first fixes a j such that $h \in \mathfrak{g}_j$, and decomposes h as $h^c + h^0$, where $h^c \in Z_{\mathfrak{g}_j}$, $h^0 \in [\mathfrak{g}_j, \mathfrak{g}_j]$. Then one considers the Jordan decomposition $h_s^0 + h_n^0$ of h^0, where $h_s^0, h_n^0 \in [\mathfrak{g}_j, \mathfrak{g}_j]$ and $\operatorname{ad} h^0 = \operatorname{ad} h_s^0 + \operatorname{ad} h_n^0$ is the Jordan decomposition of the operator

$$\operatorname{ad} h^0 : [\mathfrak{g}_j, \mathfrak{g}_j] \to [\mathfrak{g}_j, \mathfrak{g}_j].$$

The existence of the decomposition $h^0 = h_s^0 + h_n^0$ is well known (it follows for instance from [B, Ch.VII,§5, Prop.1] applied to the embedding $[\mathfrak{g}_j, \mathfrak{g}_j] \subset$

$\mathfrak{gl}([\mathfrak{g}_{j+1}, \mathfrak{g}_{j+1}])$). Finally one sets $h_s := h^c + h_s^0$, $h_n := h_n^0$. Then h_s is a well-defined semisimple element (prove this using the fact that $\mathfrak{g}_j$ is reductive in $\mathfrak{g}_{j+1}$!) and $h_s, h_n \in \mathfrak{g}$, $[h_s, h_n] = 0$. Furthermore, the key point is that the decomposition $h = h_s + h_n$ does not depend on j. This latter fact follows for instance from [B, Ch.VII,§5, Rem.2] and we recommend the reader to prove this in full detail. In addition, it is a standard property that $\operatorname{ad} h_s$ is a polynomial in $\operatorname{ad} h$ with no constant term.

If $\mathfrak{g}$ is locally reductive and $\mathfrak{h} \subset \mathfrak{g}$ is any Lie subalgebra, we can now call $\mathfrak{h}$ *splittable* if for every $h \in \mathfrak{h}$, both vectors h_s and h_n also belong to $\mathfrak{h}$ (in general $h_s, h_n \in \mathfrak{g}$). (Please read [B, Ch.VII, §5] to appreciate this definition!) Furthermore, for any subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we set

$$\overline{\mathfrak{g}_{\mathfrak{h}}^0} := \bigcap_i \mathfrak{g}_{\mathfrak{h}_i}^0,$$

where $\mathfrak{h} = \bigcup \mathfrak{h}_i$ is some exhaustion of $\mathfrak{h}$ by finite-dimensional Lie algebras $\mathfrak{h}_i$, and $\mathfrak{g}_{\mathfrak{h}_i}^0$ is as defined in (3.9). (Check that $\overline{\mathfrak{g}_{\mathfrak{h}}^0}$ does not depend on the choice of exhaustion!)

We can now prove the following general result which is a slightly stronger version of Theorem 3.2 in [DPSn].

Theorem 6.9 *Let $\mathfrak{g}$ be a locally reductive Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. The following conditions are equivalent:*

(1) $\mathfrak{h}$ is locally nilpotent and $\mathfrak{h} = C(\mathfrak{h}_s)$, where $\mathfrak{h}_s := \{h_s \mid h \in \mathfrak{h}\}$;
(2) $\mathfrak{h} = C(\mathfrak{h}_s)$ and $\mathfrak{h}_s$ is a subalgebra;
(3) $\mathfrak{h} = C(\mathfrak{t})$ for a maximal toral subalgebra $\mathfrak{t} \subset \mathfrak{g}$;
(4) $\mathfrak{h} = \overline{\mathfrak{g}_{\mathfrak{h}}^0}$.

Any subalgebra $\mathfrak{h}$ satisfying the conditions (1)–(4) is splittable and self-normalizing.

Definition 6.10 We define a *Cartan subalgebra* of $\mathfrak{g}$ to be any subalgebra $\mathfrak{h}$ satisfying the equivalent conditions *(1)–(4)* from Theorem 6.9. We call a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ *splitting* if $\mathfrak{g}$ is an $\mathfrak{h}$-weight module.

A splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ has a generalized $\mathfrak{h}$-root decomposition such that all generalized root spaces are weight spaces and the zero weight space equals $\mathfrak{h}$ (prove this!).

We start the proof of Theorem 6.9 with the following proposition.

Proposition 6.11 *Let $\mathfrak{h}$ be a locally nilpotent subalgebra of a locally reductive Lie algebra $\mathfrak{g}$. Then the following assertions hold:*

(a) $\mathfrak{h} \subseteq C(\mathfrak{h}_s)$;
(b) $\mathfrak{h}_s$ is a toral subalgebra of $\mathfrak{g}$;
(c) $C(\mathfrak{h}_s)$ is a self-normalizing subalgebra of $\mathfrak{g}$.

Proof Let $h, h' \in \mathfrak{h}$. The local nilpotence of $\mathfrak{h}$ implies $(\operatorname{ad} h)^n(h') = 0$ for some n. Since $\operatorname{ad} h_s$ is a polynomial with no constant term in $\operatorname{ad} h$, it follows that $(\operatorname{ad} h_s)(\operatorname{ad} h)^{n-1}(h') = 0$. Because an element commutes with its semisimple part, $(\operatorname{ad} h)^{n-1}(\operatorname{ad} h_s)(h') = 0$, and it follows by induction that $(\operatorname{ad} h_s)^n(h') = 0$. Hence $(\operatorname{ad} h_s)(h') = 0$. Thus $\mathfrak{h} \subseteq C(\mathfrak{h}_s)$.

Furthermore by the same argument, $(\operatorname{ad} h')(h_s) = 0$ implies that $(\operatorname{ad} h'_s)(h_s) = 0$. Therefore any two elements of $\mathfrak{h}_s$ commute. Since the sum of any two commuting semisimple elements is semisimple, $\mathfrak{h}_s$ is a subalgebra.

Finally, suppose x is in the normalizer of $C(\mathfrak{h}_s)$. For any $y \in \mathfrak{h}_s$, we have $[x, y] \in C(\mathfrak{h}_s)$. Thus $[[x, y], y] = 0$, and as y is semisimple it follows that $[x, y] = 0$. Hence, $x \in C(\mathfrak{h}_s)$, i.e., $C(\mathfrak{h}_s)$ is self-normalizing. □

Lemma 6.12 *If $\mathfrak{h}$ is locally nilpotent and splittable, then $\overline{\mathfrak{g}^0_{\mathfrak{h}}} = C(\mathfrak{h}_s)$.*

Proof By Bourbaki [B, Ch.VII, § 5, Prop. 5] $\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{h}_n$, with $\mathfrak{h}_n$ being the subalgebra of all nilpotent elements in $\mathfrak{h}$. It follows that $\overline{\mathfrak{g}^0_{\mathfrak{h}}} = \overline{\mathfrak{g}^0_{\mathfrak{h}_s}} \cap \overline{\mathfrak{g}^0_{\mathfrak{h}_n}}$. Since elements of $\mathfrak{h}_s$ are semisimple, $\overline{\mathfrak{g}^0_{\mathfrak{h}_s}} = C(\mathfrak{h}_s)$. Clearly $\overline{\mathfrak{g}^0_{\mathfrak{h}_n}} = \mathfrak{g}$. Hence $\overline{\mathfrak{g}^0_{\mathfrak{h}}} = C(\mathfrak{h}_s)$. □

Proof of Theorem 6.9 Fix an exhaustion $\mathfrak{g} = \bigcup_{i \in \mathbb{Z}_{>0}} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is a finite-dimensional reductive Lie algebra with $\mathfrak{g}_i$ reductive in $\mathfrak{g}_{i+1}$.

Clearly *(1)* implies *(2)* by Proposition 6.11 *(b)*. To show that *(2)* implies *(1)*, we need to show that any subalgebra $\mathfrak{h}$ satisfying *(2)* is locally nilpotent. The equality $\mathfrak{h} = C(\mathfrak{h}_s)$ implies that $\mathfrak{h}_s$ commutes with $\mathfrak{h}$. Now consider a general element $h = h_s + h_n \in \mathfrak{h}$. Choose k such that $(\operatorname{ad} h_n)^k = 0$. For any $x \in \mathfrak{h}$,

$$(\operatorname{ad} h)^k(x) = (\operatorname{ad}(h_s + h_n))^k(x) = (\operatorname{ad} h_n)^k(x) = 0.$$

Hence $\mathfrak{h}$ is locally nilpotent, i.e., *(2)* implies *(1)*.

Next, it is clear that *(2)* implies *(3)*. The equality $\mathfrak{h} = C(\mathfrak{h}_s)$ shows that any semisimple element of $\mathfrak{g}$ that centralizes $\mathfrak{h}_s$ is already in $\mathfrak{h}_s$. Thus $\mathfrak{h}_s$ is a maximal toral subalgebra and *(3)* holds.

Let $\mathfrak{h}$ satisfy *(3)*. We first prove that $\mathfrak{h}$ is splittable. For any $i \in \mathbb{Z}_{>0}$, note that

$$\mathfrak{h} \cap \mathfrak{g}_i = C(\mathfrak{t}) \cap \mathfrak{g}_i = \bigcap_{k \geq i} (C_{\mathfrak{g}_k}(\mathfrak{t} \cap \mathfrak{g}_k) \cap \mathfrak{g}_i),$$

where $C_{\mathfrak{g}_k}(\cdot)$ denotes the centralizer in $\mathfrak{g}_k$. Since $\dim \mathfrak{g}_i < \infty$, we have $\mathfrak{t} \cap \mathfrak{g}_i = C_{\mathfrak{g}_j}(\mathfrak{t} \cap \mathfrak{g}_j) \cap \mathfrak{g}_i$ for some sufficiently large $j \geq i$. Since $\mathfrak{t} \cap \mathfrak{g}_j$ is a subalgebra of $\mathfrak{g}_j$, we know from [B, Ch.VII, § 5, Prop. 3, Cor. 1] that $C_{\mathfrak{g}_j}(\mathfrak{t} \cap \mathfrak{g}_j)$ is a splittable subalgebra of $\mathfrak{g}_j$. The reductive Lie algebra $\mathfrak{g}_i$ is also splittable by Bourbaki [B, Ch.VII, § 5, Prop. 2]. Then the intersection $C_{\mathfrak{g}_j}(\mathfrak{t} \cap \mathfrak{g}_j) \cap \mathfrak{g}_i$ is splittable as well. Being a union of splittable algebras, $\mathfrak{h}$ is splittable.

Since $\mathfrak{h}$ is splittable and every element of $\mathfrak{h}$ commutes with $\mathfrak{t}$, $\mathfrak{h}_s$ is a subset of toral elements commuting with $\mathfrak{t}$. The maximality of $\mathfrak{t}$ implies now $\mathfrak{t} = \mathfrak{h}_s$, i.e., *(2)* follows from *(3)*.

Note that *(1)* implies *(4)*. Indeed, let $\mathfrak{h}$ satisfy *(1)*. We have already proved that then $\mathfrak{h}$ is splittable, so by Lemma 6.12, $\mathfrak{h} = C(\mathfrak{h}_s) = \overline{\mathfrak{g}^0_{\mathfrak{h}}}$.

To show that *(4)* implies *(1)*, assume that $\mathfrak{h} = \overline{\mathfrak{g}^0_{\mathfrak{h}}}$. Then clearly $\mathfrak{h}$ is locally nilpotent. An argument similar to that above shows that $\mathfrak{h}$ is also splittable. Indeed, for any $i \in \mathbb{Z}_{>0}$, we have

$$\overline{\mathfrak{g}^0_{\mathfrak{h}}} \cap \mathfrak{g}_i = \bigcap_{k \geq i} \left(\left((\mathfrak{g}_k)^0_{\mathfrak{h} \cap \mathfrak{g}_k} \right) \cap \mathfrak{g}_i \right).$$

The finite-dimensionality of $\mathfrak{g}_i$ yields $\overline{\mathfrak{g}^0_{\mathfrak{h}}} \cap \mathfrak{g}_i = (\mathfrak{g}_j)^0_{\mathfrak{h} \cap \mathfrak{g}_j} \cap \mathfrak{g}_i$ for some sufficiently large $j \geq i$. It is well known that $(\mathfrak{g}_j)^0_{\mathfrak{h} \cap \mathfrak{g}_j}$ is a splittable subalgebra of $\mathfrak{g}_j$, see [B, Ch.VII, §1, Prop. 11]. Since $\mathfrak{g}_i$ is also splittable, the intersection $(\mathfrak{g}_j)^0_{\mathfrak{h} \cap \mathfrak{g}_j} \cap \mathfrak{g}_i$ is splittable as well. Hence $\overline{\mathfrak{g}^0_{\mathfrak{h}}} \cap \mathfrak{g}_i$ is splittable. Being a union of splittable algebras, $\mathfrak{h}$ is splittable. Therefore Lemma 6.12 implies that $\mathfrak{h} = \overline{\mathfrak{g}^0_{\mathfrak{h}}} = C(\mathfrak{h}_s)$.

In addition, by Proposition 6.11 *(c)*, a subalgebra $\mathfrak{h}$ satisfying *(1)* is self-normalizing. As we have already seen that a subalgebra $\mathfrak{h}$ satisfying *(4)* is splittable, the proof of Theorem 6.9 is complete. □

6.2 Borel and Parabolic Subalgebras of $\mathfrak{sl}(\infty)$ and $\mathfrak{gl}(\infty)$

In the second part of this chapter, we present a description of arbitrary Borel and parabolic subalgebras of $\mathfrak{g} = \mathfrak{gl}(\infty)$. The discussion is brief and is based on the announcement [DP3] and the papers [D] and [DP]. The complete proofs can be found in the latter two works.

After having discussed general Cartan subalgebras, we can now rephrase our definition of a splitting Borel subalgebra of $\mathfrak{sl}(\infty)$, or of $\mathfrak{g}$ (see Chap. 5), as a Borel subalgebra (i.e., a maximal locally solvable subalgebra) that contains a splitting Cartan subalgebra. Splitting Borel subalgebras were discussed in detail in Chap. 3. It is not at all obvious whether non-splitting Borel subalgebras exist at all, but as we will now see, they do exist and are stabilizers of certain generalized flags in V.

A first thought that comes to mind when one attempts to describe general Borel subalgebras of $\mathfrak{g}$ is to try to characterize them as stabilizers of arbitrary maximal generalized flags in V, or equivalently of arbitrary maximal chains of subspaces in V. (The equivalence of these two approaches follows from the existence of the natural bijection between maximal generalized flags and maximal chains that we mentioned in Chap. 5, see [DP3].) However, somewhat surprisingly, it turns out that a given maximal locally solvable subalgebra of $\mathfrak{g}$ can in general stabilize more than

one maximal generalized flag in V (and consequently, more than one maximal chain of subspaces in V). More precisely, the following is true (and is not very difficult to prove, so the reader should try!) [DP3].

Proposition 6.13 *Any Borel subalgebra of* $\mathfrak{g}$ *is the stabilizer of a (in general not unique) maximal generalized flag in* V.

The interesting problem is whether it is possible to choose a class of generalized flags in V that is in a natural bijection with all Borel subalgebras of $\mathfrak{g}$. The answer is affirmative, and we will now introduce this class. First of all, recall that for $U \subset V$ we have $U^\perp := \{v_* \in V_* \mid v_*(U) = 0\}$, and for $W \subset V_*$ we have $W^\perp := \{v \in V \mid v(W) = 0\}$.

If U is a subspace of V, we call U *closed* (or *Mackey closed*) if $(U^\perp)^\perp = U$. Every subspace of the form $\operatorname{span}\{e_i\}_{i\in I} \subset V$, for some subset $I \subset \mathbb{Z}_{>0}$, is closed. On the other hand, the subspace $\operatorname{span}\{e_i - e_{i+1}\}_{i\in\mathbb{Z}_{>0}} \subset V$ is not closed as $(\operatorname{span}\{e_i - e_{i+1}\}_{i\in\mathbb{Z}_{>0}})^\perp = \{0\}$ and $((\operatorname{span}\{e_i - e_{i+1}\}_{i\in\mathbb{Z}_{>0}})^\perp)^\perp = V$, while $\operatorname{span}\{e_i - e_{i_i}\}_{i\in\mathbb{Z}_{>0}} \neq V$.

In [DP3] we have introduced the following notion of a closed generalized flag in V. A generalized flag $\{\mathcal{F}_k\}$ in V is *closed* if and only if for any pair of indices k', k'' (where, as in Chap. 5, k' is the immediate predecessor of k''), $(\mathcal{F}_{k''}^\perp)^\perp = \mathcal{F}_{k''}$, and $(\mathcal{F}_{k'}^\perp)^\perp = \mathcal{F}_{k'}$ or $(\mathcal{F}_{k'}^\perp)^\perp = \mathcal{F}_{k''}$. The main result of [DP3] is the following theorem.

Theorem 6.14 *Any Borel subalgebra of* $\mathfrak{g}$ *is the stabilizer of a unique maximal closed generalized flag in* V*, and this induces a bijection between the Borel subalgebras of* $\mathfrak{g}$ *and the maximal closed generalized flags in* V*. This statement holds also for* $\mathfrak{sl}(\infty)$.

The proof of Theorem 6.14 can be found in the Appendix of [D]. We restrict ourselves to a few comments.

Here is an example of a maximal closed generalized flag in V that is not maximal as a generalized flag.

Example 6.15 Consider the chain $\mathcal{F}$:

$$0 \subset V_1 \subset V_2 \subset \cdots \subset \tilde{V} \subset V, \tag{6.2}$$

where $\tilde{V} = \operatorname{span}\{e_i - e_{i+2}\}_{i\in\mathbb{Z}_{>0}}$ and $V_j = \operatorname{span}\{e_i - e_{i+2}\}_{i\le j}$ for $j = 1, 2, \ldots$. Note that $\dim V/\tilde{V} = 2$, while $\dim V_{i+1}/V_i = 1$ for $i \ge 0$. Any maximal generalized flag containing $\mathcal{F}$ is of the form $\mathcal{F} \cup \{\tilde{V}'\}$ where $\tilde{V}'$ is (any) codimension 1 subspace of V that contains $\tilde{V}$. Furthermore, $\mathcal{F}$ is a closed generalized flag, as $(V_i^\perp)^\perp = V_i$ and $(\tilde{V}^\perp)^\perp = V$. (Prove both statements!) It is clear that for no choice of $\tilde{V}'$, the generalized flag $\mathcal{F} \cup \{\tilde{V}'\}$ is a closed generalized flag as $((\tilde{V})^\perp)^\perp \neq \tilde{V}$ and $((\tilde{V})^\perp)^\perp \neq \tilde{V}'$ for any codimension 1 subspace $\tilde{V}' \subset V$. Hence $\mathcal{F}$ is a maximal closed generalized flag, but not a maximal generalized flag.

It is true that for every maximal chain of the form $\mathcal{F} \cup \{\tilde{V}'\}$, its stabilizer in $\mathfrak{g}$ coincides with the stabilizer of $\mathcal{F}$. By Theorem 6.14, the latter is a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, and it is an example of a Borel subalgebra of $\mathfrak{g}$ that is the

stabilizer of an infinite number of maximal chains of subspaces in V (as well as of an infinite number of maximal generalized flags in V). Furthermore, this implies that $\mathfrak{b}$ is a non-splitting Borel subalgebra since, as shown in Chap. 5, a splitting Borel subalgebra stabilizes a unique maximal generalized flag in V. ◯

In what follows $\mathfrak{n}_{\mathfrak{g}} \subset \mathfrak{g}$ will denote the set of all nilpotent endomorphisms in $\mathfrak{g} = V \otimes V_*$. The following result is from [DP3].[1]

Proposition 6.16 *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Set $\mathfrak{n}_{\mathfrak{b}} := \mathfrak{n}_{\mathfrak{g}} \cap \mathfrak{b}$.*

(1) $\mathfrak{n}_{\mathfrak{b}}$ is an ideal in $\mathfrak{b}$ and $\mathfrak{b} = N(\mathfrak{n}_{\mathfrak{b}})$.
(2) there exists a toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}_{\mathfrak{b}}$.

Proposition 6.16 is analogous to the corresponding statement in the finite-dimensional case; however, *(2)* differs in that $\mathfrak{t}$ need not be a maximal toral subalgebra. Moreover, even if $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$, $\mathfrak{t}$ does not have to be splitting. For instance, if $\mathfrak{b}$ is the non-splitting Borel subalgebra that stabilizes the maximal generalized flag (6.2), $\mathfrak{t}$ can be chosen as the maximal toral subalgebra with maximal dual system

$$V_i := \operatorname{span}\{e_i - e_{i+2}\}, \ (V_*)_i := \operatorname{span}\left\{ \sum_{1 \le j \le i, i-j \equiv 0 (\mathrm{mod}\, 2)} e_j^* \right\}. \tag{6.3}$$

This maximal toral subalgebra is a Cartan subalgebra, but it is not splitting.

Another new phenomenon is that a Borel subalgebra can contain maximal toral subalgebras $\mathfrak{l}$ of $\mathfrak{g}$ that do not satisfy *(2)*, i.e., for which $\mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{b}}$ is a proper subalgebra of $\mathfrak{b}$. For instance, let $\mathfrak{b} \subset \mathfrak{g}$ be the Borel subalgebra defined as the stabilizer of the following maximal flag in V: $\cdots \subset V_2 \subset V_1 \subset V$ where $V_i = \operatorname{span}\{e_j\}_{j>i}$. This flag is of course compatible with the basis $\{e_i\}$, hence $\mathfrak{b}$ is splitting. On the other hand, if $\mathfrak{l} = \mathfrak{t}$ is the maximal toral subalgebra with maximal dual system (6.3), a straightforward computation (do it!) shows that each V_i is $\mathfrak{l}$-stable, hence $\mathfrak{l} \subset \mathfrak{b}$. However, $\mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{b}} \neq \mathfrak{b}$, as for instance $e_1 \otimes e_1^* \in \mathfrak{b}$, but $e_1 \otimes e_1^* \notin \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{b}}$. (Check this directly!)

Here is an extreme example. We will write down a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{b} = \mathfrak{n}_{\mathfrak{b}}$. In this case, Proposition 6.16 *(2)* holds with $\mathfrak{t} = \{0\}$.

Example 6.17 Let $\{v_q\}_{q \in \mathbb{Q}}$ be a basis of V ordered by the elements of $\mathbb{Q}$ as in Example 5.5. Consider the subspace $\tilde{V}_*$ of V^* spanned by the linear functions $\tilde{v}_q : V \to \mathbb{C}$ defined as follows:

$$\tilde{v}_q(v_r) = \begin{cases} 1 \text{ for } q < r \\ 0 \text{ for } q \ge r. \end{cases} \tag{6.4}$$

[1] The definition of $\mathfrak{n}_{\mathfrak{b}}$ given in [DP3] should be replaced with the definition given here.

The tensor product $V \otimes \tilde{V}_*$ is an associative algebra isomorphic to the associative algebra $V \otimes V_*$. This fact is nontrivial and has been established by G. Mackey [Mac]. More generally, Mackey shows that if U and $\tilde{U}$ are any countable-dimensional vector spaces with nondegenerate pairing $U \times \tilde{U} \to \mathbb{C}$, the tensor product $U \otimes \tilde{U}$ is isomorphic as an associative algebra to the standard tensor product $\mathrm{span}\{e_i\}_{i\in\mathbb{Z}_{>0}} \otimes \mathrm{span}\{e_i^*\}_{i\in\mathbb{Z}_{>0}}$, i.e., to $V \otimes V_*$. The same result is also proved in [NP].

We can now identify $\mathfrak{g}$ with $V \otimes \tilde{V}_*$ considered as a Lie algebra. Let $\mathcal{F}$ be the maximal closed generalized flag in V defined by $\mathcal{F} = \{F_{q'}, F_{q''}\}_{q\in\mathbb{Q}}$, where

$$F_{q'} := \mathrm{span}\{v_r\}_{r<q}, \; F_{q''} := \mathrm{span}\{v_r\}_{r\le q}.$$

Using (6.4) one verifies that $(F_{q'}^{\perp})^{\perp} = F_{q''}$ and $(F_{q''}^{\perp})^{\perp} = F_{q''}$ for every $q \in \mathbb{Q}$, hence $\mathcal{F}$ is a maximal closed generalized flag. Furthermore, one can show that $\mathfrak{b} = \mathfrak{n}_{\mathfrak{b}}$, where $\mathfrak{b}$ is the stabilizer of $\mathcal{F}$ in $\mathfrak{g}$. In this way, $\mathfrak{b}$ is a maximal locally solvable subalgebra which, in addition, is locally nilpotent! In particular, $\mathfrak{b} \subset \mathfrak{sl}(\infty)$. ◯

Finally, we can state a description of general parabolic subalgebras of the Lie algebras $\mathfrak{g} = \mathfrak{gl}(\infty)$ and $\mathfrak{sl}(\infty)$. By definition, a parabolic subalgebra of $\mathfrak{g}$ is an arbitrary subalgebra of $\mathfrak{g}$ that contains a Borel subalgebra of $\mathfrak{g}$. The description of parabolic subalgebras is stated in terms of a generalization of the notion of a closed generalized flag. We define a generalized flag $\mathcal{F}$ in V or V_* to be *semiclosed* if, for any pair of indices k', k'' such that k' is the immediate predecessor of k'', $(\mathcal{F}_{k'}^{\perp})^{\perp} = \mathcal{F}_{k'}$ or $(\mathcal{F}_{k'}^{\perp})^{\perp} = \mathcal{F}_{k''}$. We denote the stabilizer of $\mathcal{F}$ in $\mathfrak{g}$ by $\mathrm{St}_{\mathcal{F}}$. A pair of semiclosed generalized flags $\mathcal{F}$ in V and $\mathcal{G}$ in V_* is called *taut* if $\mathcal{F}$ is $\mathrm{St}_{\mathcal{G}}$-stable and $\mathcal{G}$ is $\mathrm{St}_{\mathcal{F}}$-stable. The following theorem is proved in [DP].

Theorem 6.18 *For any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g} = V \otimes V_*$, there is a unique taut pair of semiclosed generalized flags $\mathcal{F}$ in V and $\mathcal{G}$ in V_*, and a toral subalgebra $\mathfrak{t}$ lying in $\mathrm{St}_{\mathcal{F}} \cap \mathrm{St}_{\mathcal{G}}$, such that $\mathfrak{p}$ is an ideal in $\mathrm{St}_{\mathcal{F}} \cap \mathrm{St}_{\mathcal{G}}$ and $\mathfrak{p} \dotplus \mathfrak{t} = \mathrm{St}_{\mathcal{F}} \cap \mathrm{St}_{\mathcal{G}}$. For the case that $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{sl}(\infty)$, $\mathrm{St}_{\mathcal{F}} \cap \mathrm{St}_{\mathcal{G}}$ should be replaced by $\mathrm{St}_{\mathcal{F}} \cap \mathrm{St}_{\mathcal{G}} \cap \mathfrak{sl}(\infty)$ in the above statements.*

Somewhat informally, Theorem 6.18 states that "up to a toral subalgebra $\mathfrak{t}$" $\mathfrak{p}$ equals the joint stabilizer of a unique taut pair of semiclosed generalized flags in V and V_*. An almost trivial, but nevertheless truly infinite-dimensional example is the parabolic subalgebra $\mathfrak{p} = \mathfrak{sl}(\infty)$ of $\mathfrak{g} = \mathfrak{gl}(\infty)$. (In the finite-dimensional case $\mathfrak{sl}(n)$ is not a parabolic subalgebra of $\mathfrak{gl}(n)$!) Here the unique pair of semiclosed generalized flags is the pair $\{0\} \subset V$, $\{0\} \subset V_*$, and $\mathfrak{t}$ can be chosen to equal the span of any semisimple element of $\mathfrak{gl}(\infty)$ with nonzero trace. Another example is of course any Borel subalgebra of $\mathfrak{g}$. Here $\mathcal{F}$ equals any maximal closed generalized flag in V, $\mathfrak{b}$ equals $\mathrm{St}_{\mathcal{F}}$, and we claim that $\mathcal{G}$ is determined uniquely by the condition that the pair $\mathcal{F}, \mathcal{G}$ be taut. This follows from [DP3]. Moreover, we have the following.

Lemma 6.19 *Any maximal semiclosed generalized flag in V or V_* is closed.*

The proof of Lemma 6.19 is an exercise for the reader (alternative: look it up in [DP]). Lemma 6.19 is in complete agreement with the fact (following directly from the definitions) that a Borel subalgebra is nothing but a minimal parabolic subalgebra.

We strongly suggest to the reader to work out some nontrivial examples of parabolic subalgebras. For instance, a parabolic subalgebra can be locally reductive. Can you find a criterion on a taut pair of semiclosed generalized flags in V and V_* for this to be the case?

The case of a general finitary root-reductive Lie algebras, and in particular of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$, is considered in [DP]. Here all statements are similar to the ones given above, except that the notion of a semiclosed generalized flag should be replaced by a semiclosed isotropic generalized flag. Trying to find the exact statements and their proofs provides a wide possibility of exercises for the reader. One can then always look up the result in [DP] or in [D], where the case of Borel subalgebras of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ is worked out.

6.3 Closing Remarks

Several comments are in order.

1. We presented above an explicit description of the maximal toral subalgebras (and their centralizers) of $\mathfrak{g} = \mathfrak{gl}(\infty)$. This description carries over to $\mathfrak{sl}(\infty)$ without changes. In [DPSn] the reader will find descriptions of the maximal toral subalgebras of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. Their centralizers (for $\mathfrak{o}(\infty)$) are in general no longer commutative, but are always nilpotent. This leads to the following result, [DPSn]: any Cartan subalgebra of a root-reductive Lie algebra is nilpotent in a maximum of two steps. No explicit description of arbitrary maximal toral subalgebras or Cartan subalgebras of non-root-reductive locally reductive Lie algebras is known.
2. Every splitting Cartan subalgebra is maximal toral, but a maximal toral Cartan subalgebra is not necessarily splitting. Indeed, the maximal dual system (6.1) yields a non-splitting Cartan subalgebra. Here is a simpler example: for $i \geq 1$,

$$V^i := \operatorname{span}\{e_i - e_{i+1}\}, \quad V_*^i := \operatorname{span}\left\{\sum_{j=1}^{i} e_j^*\right\} \tag{6.5}$$

   corresponds to a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{h} = \mathfrak{h}_s$, $V_* = V'_*$, $\dim V/V' = 1$, and is not splitting.
3. If $\mathfrak{h} \subset \mathfrak{g}$ is a toral subalgebra such that $\mathfrak{g}$ admits an $\mathfrak{h}$-root decomposition, then $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$. This follows from condition *(2)* of Theorem 6.9 since in this case $\mathfrak{h} = C(\mathfrak{h})$.

4. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{gl}(\infty)$ is splitting if and only if $\mathfrak{g}$ is a locally $\mathfrak{h}$-finite module, or equivalently if both V and V_* are locally $\mathfrak{h}$-finite modules. Moreover, if $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h} = \overline{\mathfrak{g}^0_{\mathfrak{h}}}$ being the zero weight space in $\mathfrak{g}$ implies that $\mathfrak{h}$ equals $\mathfrak{g}^0_{\mathfrak{h}}$. Consequently, splitting Cartan subalgebras are precisely the subalgebras $\mathfrak{h}$ that were considered in Chaps. 3–5; in other words, any splitting Cartan subalgebra can be identified with our fixed subalgebra $\mathfrak{h}$ and can be used to define the decomposition (5.1). We will address the issue of conjugacy of splitting Cartan subalgebras in Sect. 7.2 below.
5. Our considerations in Chap. 4 imply that the Lie algebra $\mathfrak{sl}(\infty)$ is not isomorphic to $\mathfrak{sl}(i, \mathrm{ad})$ for any $i \geq 2$, since $\mathrm{Stz}(\mathfrak{sl}(\infty)) = \mathbb{Z}_{>0}$ while $\mathrm{Stz}(\mathfrak{sl}(i, \mathrm{ad})) \neq \mathbb{Z}_{>0}$ (more generally one can replace $\mathfrak{sl}(i, \mathrm{ad})$ by any locally simple Lie subalgebra $\mathfrak{s}$ with $\mathrm{Stz}(\mathfrak{s}) \neq \mathbb{Z}_{>0}$). We can now give another proof of the same fact based on the theory of Cartan subalgebras of $\mathfrak{sl}(\infty)$. Indeed, the results in this chapter imply that for maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{sl}(\infty)$, $\mathfrak{t}$ is contained properly in the unique maximal locally $\mathfrak{t}$-finite $\mathfrak{t}$-submodule $\mathfrak{sl}(\infty)' \subset \mathfrak{sl}(\infty)$. On the other hand, let $\tilde{\mathfrak{t}}$ be a maximal toral subalgebra of $\mathfrak{sl}(i, \mathrm{ad})$ constructed via induction by extending a Cartan subalgebra $\mathfrak{h}_i$ of $\mathfrak{sl}(i)$ to a Cartan subalgebra $\mathfrak{h}_{i^2-1}$ of $\mathfrak{sl}(i^2 - 1)$, etc. Then the reader will check immediately that $\tilde{\mathfrak{t}} = \mathfrak{sl}(i, \mathrm{ad})'$, where $\mathfrak{sl}(i, \mathrm{ad})'$ stands for the unique maximal locally $\tilde{\mathfrak{t}}$-finite $\mathfrak{t}$-submodule of $\mathfrak{sl}(i, \mathrm{ad})$. Hence $\mathfrak{sl}(\infty)$ and $\mathfrak{sl}(i, \mathrm{ad})$ are not isomorphic as Lie algebras. We urge the reader to find a third proof of this fact (use one more time the idea in the proof of the fact that $\mathfrak{sl}(\infty)$ and $\mathfrak{sp}(\infty)$ are not isomorphic, Chap. 3).
6. An important structural problem in the theory of Cartan subalgebras is to describe the conjugacy classes of all Cartan subalgebras of $\mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. More precisely, let $\mathfrak{g} = \mathfrak{gl}(\infty)$ and let $GL(V, V_*)$ denote the group of automorphisms of V that keep V_* fixed as a subspace of the dual space V^*. Then, clearly the following four dimensions: $\dim V/V'$, $\dim(V')^{\perp}$, $\dim V_*/V'_*$, $\dim(V'_*)^{\perp}$, (here $(\,)^{\perp}$ stands for orthogonal with respect to the pairing $V \otimes V_* \to \mathbb{C}$) are invariants of the $GL(V, V_*)$-conjugacy classes of Cartan subalgebras. However, this is not a complete system of invariants. The reader should prove this by considering examples (see also [NP, Proposition 3.11]). The complete system of $GL(V, V_*)$-invariants in all four cases ($\mathfrak{g} = \mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$) was found in [DPSn]. Since the answer is quite technical we will not present it here, but will refer the reader directly to [DPSn].
7. We conclude this chapter with a wide open problem in the representation theory of $\mathfrak{g} = \mathfrak{gl}(\infty)$ (as well as of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$). It can vaguely be stated as: understanding the role played by the non-splitting Cartan, Borel and parabolic subalgebras in representation theory. More precisely, we can ask, for instance, whether every Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, defined by a one-dimensional representation $\mathbb{C}_\lambda$ of an arbitrary Borel subalgebra $\mathfrak{b}$, has a unique maximal submodule. If the answer is negative, what would be a necessary and sufficient condition for the existence of a unique maximal submodule (we know examples of non-splitting Borel subalgebras for which the answer is positive)? Dependent on these answers, one should be able to make some further steps toward the vague but interesting problem stated above.

Part II
Modules over Locally Reductive Lie Algebras

This part of the book is devoted to the theory of representations of locally reductive Lie (super)algebras, as well as of finite-dimensional Lie algebras.

Chapter 7 introduces a basic category of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$-modules called tensor modules. This category is a most natural analog of the category of finite-dimensional $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$, $\mathfrak{sp}(2n)$-modules. A main point is that the latter category is semisimple while the former is not. Tensor modules of the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ are subquotients of tensor products $V^{\otimes p} \otimes V_*^{\otimes q}$, V and V_* being the natural and conatural module, respectively ($V = V_*$ for $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$). The category of tensor modules over $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ has simple objects parameterized by pairs of Young diagrams, and it has enough injectives but not projectives. We compute the injective hulls of simple objects and show that the category is Koszul in the sense of Beilinson–Ginzburg–Soergel. We conclude by presenting a result of Vera Serganova claiming that the category of tensor modules over a classical locally reductive Lie superalgebra is equivalent to a category of tensor modules over a Lie algebra.

In Chap. 8 we study weight modules, that is, possibly infinite-dimensional modules over a locally reductive Lie algebra on which a splitting Cartan subalgebra acts semisimply. We start with some general facts about a locally finite action, and introduce the Fernando–Kac subalgebra of a module. We then introduce the shadow decomposition of a weight module of a finite-dimensional reductive Lie algebra, in particular, recalling results of Suren Fernando and Vyacheslav Futorny. We also reproduce Olivier Mathieu's notion of a coherent family of weight modules.

In the second part of Chap. 8, we pass to infinite-dimensional weight modules over a root-reductive Lie algebra. Here we prove structure results for such integrable modules, and present a recent classification of the simple bounded weight modules over the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. We then study simple highest weight $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$-modules with finite-dimensional weight spaces, and finish the chapter with a brief survey of various categories of weight modules.

Chapter 9 addresses a larger class of modules, namely $(\mathfrak{g}, \mathfrak{k})$-modules, where $\mathfrak{k}$ is a subalgebra of a locally reductive Lie algebra $\mathfrak{g}$. If $\mathfrak{k}$ is a Cartan subalgebra, $(\mathfrak{g}, \mathfrak{k})$-modules are a generalization of weight modules. Classically, $(\mathfrak{g}, \mathfrak{k})$-modules

have been studied extensively under the assumption that $\mathfrak{g}$ is a finite-dimensional reductive Lie algebra and $\mathfrak{k}$ is a symmetric subalgebra, and these $(\mathfrak{g}, \mathfrak{k})$-modules are known as Harish-Chandra modules. Much less is known when $\mathfrak{k}$ is a general reductive in $\mathfrak{g}$ Lie subalgebra. We call a $(\mathfrak{g}, \mathfrak{k})$-module a generalized Harish-Chandra module if it satisfies the additional condition of being of finite type, or admissible over $\mathfrak{k}$. This condition is automatic for simple $(\mathfrak{g}, \mathfrak{k})$-modules if $\mathfrak{k}$ is symmetric.

Our main tool to study the generalized Harish-Chandra modules is the Zuckerman functor. For a finite-dimensional reductive Lie algebra, we present a reconstruction theorem claiming that certain simple $(\mathfrak{g}, \mathfrak{k})$-modules are necessarily obtained via the Zuckerman functor. Under the addition condition that the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2)$, we also state an equivalence of a category of $(\mathfrak{g}, \mathfrak{sl}(2))$-modules with a "highest weight type" category of $\mathfrak{g}$-modules. We conclude Chap. 9 with a construction of generalized Harish-Chandra modules for certain locally simple infinite-dimensional Lie algebras. The main feature here is to ensure that these modules are indeed of finite type over the respective subalgebra $\mathfrak{k}$.

Chapter 7
Tensor Modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$

In this chapter we study modules over the infinite-dimensional classical Lie algebras $\mathfrak{g} = \mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ that are obtained by taking tensor products of the natural and conatural module of $\mathfrak{g}$. We will examine the structure of these modules and realize them in a suitable category $\mathbb{T}_{\mathfrak{g}}$. This category is not semisimple, but is Koszul in the sense of Beilinson–Ginzburg–Soergel.

7.1 Structure of Mixed Tensor Products

It is natural to ask about the structure of the $\mathfrak{g}$-modules

$$T^{p,q} := V^{\otimes p} \otimes V_*^{\otimes q}$$

where V is the natural module and V_* is the conatural module.

Let's first consider the case when $\mathfrak{g}$ is one of the finite-dimensional classical Lie algebras $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{o}(n)$, $\mathfrak{sp}(2n)$. Except for $\mathfrak{g} = \mathfrak{o}(n)$, which has also spin representations, all irreducible finite-dimensional $\mathfrak{g}$-modules occur as subquotients of $T^{p,q}$, and in fact it is enough to consider just $T^{p,0}$, since $V^* \simeq \Lambda^{n-1} V$ for $\mathfrak{g} = \mathfrak{sl}(n)$ and $V^* = V$ for $\mathfrak{g} = \mathfrak{o}(n)$, $\mathfrak{sp}(2n)$ [Wey]. The latter claim does not hold for $\mathfrak{g} = \mathfrak{gl}(n)$; for example, V^* does not occur as a subquotient of $T^{p,0}$. However, one can obtain all irreducible finite-dimensional $\mathfrak{gl}(n)$-modules with integral weights as subquotients of $T^{p,q}$.

One motivation for studying the $\mathfrak{g}$-modules $T^{p,q}$ for $\mathfrak{g} = \mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ is to understand the stable properties of finite-dimensional modules as n goes to infinity. We actually gain additional information because $T^{p,q}$ is no longer semisimple.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_7

Let us begin by recalling the decomposition of the module $V^{\otimes p}$ given by Schur–Weyl duality in the case that $\mathfrak{g} = \mathfrak{gl}(n)$ or $\mathfrak{sl}(n)$. For this purpose, we need the following definitions.

A *partition* λ is by definition a finite nonstrictly decreasing sequence of positive integers:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t).$$

For any partition λ, we set $|\lambda| := \lambda_1 + \cdots + \lambda_t$, and for our purposes it is convenient to set $\lambda_i = 0$ for $i > t$. Moreover, we identify Young diagrams with partitions by recording the row lengths from top to bottom. For the language of Young diagrams and Young tableaux, we refer the reader to [FH].

A *standard Young tableau* $\boldsymbol{\lambda}$ is obtained from a Young diagram λ by filling each box with one of the numbers $1, 2, \ldots, p = |\lambda|$ in such a way that the entries in each row and in each column are strictly increasing. Given a standard Young tableau $\boldsymbol{\lambda}$ of size p, we can define the subgroup $G_{\boldsymbol{\lambda}}^R$ (respectively, $G_{\boldsymbol{\lambda}}^C$) of S_p to be the set of all elements that preserve the rows (respectively, columns) of λ. The Young symmetrizer corresponding to a standard Young tableau of shape $\boldsymbol{\lambda}$ is the element $c_{\boldsymbol{\lambda}}$ of the group ring $\mathbb{C}[S_p]$ defined by $c_{\boldsymbol{\lambda}} := a_{\boldsymbol{\lambda}} b_{\boldsymbol{\lambda}}$, where

$$a_{\boldsymbol{\lambda}} = \sum_{g \in G_{\boldsymbol{\lambda}}^R} g \quad \text{and} \quad b_{\boldsymbol{\lambda}} = \sum_{g \in G_{\boldsymbol{\lambda}}^C} \operatorname{sgn}(g) g.$$

Irreducible representations of the symmetric group S_p are parameterized by partitions λ satisfying $|\lambda| = p$, and can be realized inside the regular representation $\mathbb{C}[S_p]$. In what follows, for any partition λ we fix a standard Young tableau $\boldsymbol{\lambda}$ by filling the Young tableau row-by-row beginning from the top and going from left to right, which we think of as the canonical standard Young tableau corresponding to λ. We denote

$$Y_\lambda := \mathbb{C}[S_p]\, c_{\boldsymbol{\lambda}},$$

where $c_{\boldsymbol{\lambda}} \in \mathbb{C}[S_p]$ is the Young symmetrizer corresponding to $\boldsymbol{\lambda}$. The left regular action makes Y_λ an irreducible S_p-module, and up to isomorphism any irreducible representation of S_p arises through this construction.

For any vector space V, the symmetric group S_p acts on $V^{\otimes p}$ by permuting the tensor factors. For any partition λ with $|\lambda| = p$, we denote

$$\mathbb{S}_\lambda(V) := \operatorname{im}\Big(c_{\boldsymbol{\lambda}} : V^{\otimes p} \to V^{\otimes p} \Big);$$

the *Schur functor* corresponding to λ sends V to $\mathbb{S}_\lambda(V)$. We set $V_\lambda := \mathbb{S}_\lambda(V)$ and $(V_*)_\lambda := \mathbb{S}_\lambda(V_*)$. (As an exercise, show that if $\lambda = (1, 1, \ldots, 1)$, then $V_\lambda = \Lambda^p(V)$, while if $\lambda = (p)$, then $V_\lambda = S^p(V)$. What is Y_λ in each of these cases?) Note that

since the actions of $\mathfrak{g}$ and S_p on $V^{\otimes p}$ (as well as on $V_*^{\otimes p}$) commute, the spaces V_λ and $(V_*)_\lambda$ are $\mathfrak{g}$-modules.

The following classical result gives a complete and elegant description of the $\mathfrak{g} \times S_p$-module structure of the modules $V^{\otimes p}$ and $V_*^{\otimes p}$.

Theorem 7.1 (Schur–Weyl Duality) *Let $\mathfrak{g} = \mathfrak{gl}(n)$ or $\mathfrak{g} = \mathfrak{sl}(n)$. Then there are isomorphisms of $\mathfrak{g} \times S_p$-modules*

$$V^{\otimes p} \simeq \bigoplus_{|\lambda|=p} V_\lambda \otimes Y_\lambda \quad \text{and} \quad V_*^{\otimes p} \simeq \bigoplus_{|\lambda|=p} (V_*)_\lambda \otimes Y_\lambda, \tag{7.1}$$

where λ runs over the set of partitions of size p with at most n rows, and the $\mathfrak{g}$-module V_λ and the S_p-module Y_λ defined above are both irreducible.

Now if $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{g} = \mathfrak{sl}(\infty)$, then the natural module V and its restricted dual V_* are clearly irreducible $\mathfrak{g}$-modules, and the classical Schur–Weyl duality simplifies as formulas (7.1) hold now for all partitions of size λ without a limit on the number of rows in λ (see e.g., [PSty]). In particular, the $\mathfrak{g}$-module $V^{\otimes n}$ is completely reducible, and the composition multiplicity of the irreducible module V_λ in $V^{\otimes n}$ is equal to the dimension of the S_p-module Y_λ (which equals the number of standard Young tableaux corresponding to the partition λ).

In contrast to the finite-dimensional case, the representation of $\mathfrak{g}$ on the space of mixed tensors $V^{\otimes p} \otimes V_*^{\otimes q}$ is not completely reducible. For example, if $\mathfrak{g} = \mathfrak{gl}(\infty)$, then the adjoint module $\mathfrak{gl}(\infty) = V \otimes V_*$ contains the codimension one submodule $\mathfrak{sl}(\infty)$, but no complement submodule. One would thus like to understand "how reducible" are the modules $T^{p,q} = V^{\otimes p} \otimes V_*^{\otimes q}$. This was accomplished in [PSty], where a description of the $\mathfrak{g}$-module structure of $T^{p,q}$ is given in terms of Littlewood–Richardson coefficients and a natural filtration of the module, namely the socle filtration.

The *socle* of a $\mathfrak{g}$-module M, denoted $\operatorname{soc} M$, is the largest semisimple submodule of M. The *socle filtration* of M is the filtration of $\mathfrak{g}$-modules

$$0 \subset \operatorname{soc} M = \operatorname{soc}^0 M \subset \operatorname{soc}^1 M \subset \cdots \subset \operatorname{soc}^i M \subset \ldots,$$

where $\operatorname{soc}^i M = p_i^{-1}(\operatorname{soc}(M/(\operatorname{soc}^{i-1} M)))$ and $p_i : M \to M/(\operatorname{soc}^{i-1} M)$ is the natural projection. The semisimple modules

$$\overline{\operatorname{soc}}^i M := (\operatorname{soc}^i M)/(\operatorname{soc}^{i-1} M)$$

are called the *layers* of the socle filtration, where by convention $\operatorname{soc}^{-1} M := \{0\}$. We say that the socle filtration of M is *exhaustive* if $M = \varinjlim(\operatorname{soc}^i M)$. We say that M has *finite Loewy length* if the socle filtration of M is finite and exhaustive. The *Loewy length* of M equals $\ell + 1$ where $\ell = \min\{r \mid \operatorname{soc}^r(M) = M\}$.

The $\mathfrak{g}$-module structure of the module $T^{p,q} = V^{\otimes p} \otimes V_*^{\otimes q}$ is given by the following theorem from [PSty].

Theorem 7.2 *Let $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then*

$$V^{\otimes p} \otimes V_*^{\otimes q} \simeq \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} \tilde{V}_{\lambda,\mu} \otimes (Y_\lambda \boxtimes Y_\mu),$$

where λ and μ are partitions, each $\tilde{V}_{\lambda,\mu} := V_\lambda \otimes (V_*)_\mu$ is an indecomposable $\mathfrak{g}$-module with irreducible socle $V_{\lambda,\mu}$, and $Y_\lambda \boxtimes Y_\mu$ is the outer tensor product of irreducible S_p- and S_q-modules.

The socle filtration of $\tilde{V}_{\lambda,\mu}$ has the following layers:

$$\overline{\mathrm{soc}}^k \tilde{V}_{\lambda,\mu} = \bigoplus_{|\gamma|=k} N^\lambda_{\gamma,\lambda'} N^\mu_{\gamma,\mu'} V_{\lambda',\mu'}, \tag{7.2}$$

where $N^\lambda_{\gamma,\lambda'}$ denotes the Littlewood–Richardson coefficients determined by the relation $s_\lambda\, s_\mu = \sum_\nu N^\nu_{\lambda,\mu}\, s_\nu$ for the Schur symmetric polynomials s_λ, s_μ, s_ν.

Note that the simple $\mathfrak{g}$-modules $V_{\lambda,\mu}$ are defined as $\operatorname{soc} \tilde{V}_{\lambda,\mu}$! Of course, for this definition to work, one should first show that $\operatorname{soc} \tilde{V}_{\lambda,\mu}$ is a simple $\mathfrak{g}$-module. Another corollary of Theorem 7.2 is that any simple subquotient of $T^{p,q}$ is actually isomorphic to a simple submodule of $T^{s,t}$ for some $s \le p$ and $t \le q$, and moreover, simple submodules of $T^{p,q}$ are parameterized by two Young diagrams when $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{g} = \mathfrak{sl}(\infty)$.

We will explain the proof of Theorem 7.2, modulo a certain technical lemma whose proof can be found in the appendix of [PSty].

Schur–Weyl duality for $\mathfrak{gl}(\infty)$ implies that $T^{p,q}$ decomposes as

$$T^{p,q} \simeq \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} (V_\lambda \otimes (V_*)_\mu) \otimes (Y_\lambda \boxtimes Y_\mu).$$

The $\mathfrak{gl}(\infty)$-module $\tilde{V}_{\lambda,\mu} := V_\lambda \otimes (V_*)_\mu$ is neither simple nor semisimple when both $\lambda \neq \emptyset$ and $\mu \neq \emptyset$, so we would like to find its socle filtration. Since $\tilde{V}_{\lambda,\mu}$ is realized as a direct summand of the $\mathfrak{gl}(\infty)$-module $T^{p,q}$, we have

$$\operatorname{soc}^k \tilde{V}_{\lambda,\mu} = \tilde{V}_{\lambda,\mu} \cap \operatorname{soc}^k T^{p,q},$$

and so we will begin by finding the socle filtration of $T^{p,q}$.

The socle of $T^{p,q}$ can be realized as the intersection of the kernels of the following contraction maps. For any pair of indices $I = (i, j)$ with $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$, we define the contraction map

$$\Phi_{i,j} : T^{p,q} \to T^{p-1,q-1} \tag{7.3}$$

by sending the element $v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^*$ to

$$\langle v_j^*, v_i \rangle \, v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes \hat{v}_j^* \otimes \cdots \otimes v_q^*.$$

To simplify notation, we will also denote $\Phi_{i,j}$ by Φ_I for a pair $I = (i, j)$. Consider the $\mathfrak{gl}(\infty) \times (S_p \times S_q)$-submodule $T^{\{p,q\}}$ of $T^{p,q}$, defined by

$$T^{\{p,q\}} := \bigcap_I \ker\Big(\Phi_I : T^{p,q} \to T^{p-1,q-1} \Big).$$

We set $V_{\lambda,\mu} := \tilde{V}_{\lambda,\mu} \cap T^{\{p,q\}}$.

The following theorem from [PSty] shows that $T^{\{p,q\}}$ is a completely reducible submodule of $T^{p,q}$, and hence is contained in the socle of $T^{p,q}$.

Theorem 7.3 *For any $p, q \in \mathbb{Z}_{\geq 0}$, there is an isomorphism of $\mathfrak{gl}(\infty) \times (S_p \times S_q)$-modules*

$$T^{\{p,q\}} \simeq \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} V_{\lambda,\mu} \otimes (Y_\lambda \boxtimes Y_\mu). \tag{7.4}$$

For any partitions λ, μ, the $\mathfrak{gl}(\infty)$-module $V_{\lambda,\mu}$ is irreducible, and is a highest weight module of highest weight $\chi := \sum_{i \in \mathbb{Z}_{>0}} \lambda_i \, \varepsilon_{2i-1} - \sum_{i \in \mathbb{Z}_{>0}} \mu_i \, \varepsilon_{2i}$ with respect to the perfect Borel subalgebra $\mathfrak{b}$ from Example 5.4. Furthermore, $V_{\lambda,\mu}$ is irreducible when restricted to $\mathfrak{sl}(\infty)$.

Proof Fix the strict linear order (5.4) on $\mathbb{Z}_{>0}$. Let $\mathfrak{g}_n$ denote the Lie subalgebra of $\mathfrak{g} = \mathfrak{gl}(\infty)$ generated by $\{E_{i,j}\}_{1 \leq i,j \leq n}$, and set $\mathfrak{b}_n := \mathfrak{b} \cap \mathfrak{g}_n$, $\mathfrak{h}_n := \mathfrak{h}_\mathfrak{g} \cap \mathfrak{g}_n$ where $\mathfrak{h}_\mathfrak{g}$ is the diagonal Cartan subalgebra of $\mathfrak{g}$. It is clear that $\mathfrak{g}_n \simeq \mathfrak{gl}(n)$, and that $\mathfrak{h}_n$ (respectively, $\mathfrak{b}_n$) is a Cartan (respectively, Borel) subalgebra of $\mathfrak{g}_n$. Moreover, the inclusion $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_{n+1}$ restricts to inclusions $\mathfrak{h}_n \hookrightarrow \mathfrak{h}_{n+1}$ and $\mathfrak{b}_n \hookrightarrow \mathfrak{b}_{n+1}$. The natural representation of $\mathfrak{g}$ has $\mathfrak{b}$-stable flag

$$V_1 \subset V_2 \subset V_3 \subset \cdots \subset V,$$

where V_n is the natural representation of $\mathfrak{g}_n$.

The modules $T_n^{p,q} := (V_n)^{\otimes p} \otimes (V_n^*)^{\otimes q}$ and the contractions $\Phi_I^{(n)} : T_n^{p,q} \to T_n^{p-1,q-1}$ are defined analogously to their infinite-dimensional counterparts. We set

$$T_n^{\{p,q\}} := \bigcap_I \ker \Phi_I^{(n)} \qquad \text{and} \qquad (V_n)_{\lambda,\mu} := T_n^{\{p,q\}} \cap (\tilde{V}_n)_{\lambda,\mu}.$$

A version of the finite-dimensional Weyl construction implies that for $n \geq p+q$ the $\mathfrak{gl}_n$-module $(V_n)_{\lambda,\mu}$ is an irreducible highest weight module with highest weight χ, regarded by restriction as a functional on $\mathfrak{h}_n$ (see the appendix of [PSty] for more

details). Furthermore, we have $\mathfrak{g}_n \times (S_p \times S_q)$-module isomorphisms

$$T_n^{\{p,q\}} \simeq \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} (V_n)_{\lambda,\mu} \otimes (Y_\lambda \boxtimes Y_\mu). \tag{7.5}$$

It is clear from the definitions that we have the following diagram of inclusions:

$$\begin{array}{ccccccccccc} T_{p+q}^{\{p,q\}} & \subset & \dots & \subset & T_n^{\{p,q\}} & \subset & T_{n+1}^{\{p,q\}} & \subset & \dots & \subset & \bigcup_{n\in\mathbb{Z}_{>0}} T_n^{\{p,q\}} = T^{\{p,q\}} \\ \cup & & & & \cup & & \cup & & & & \cup \\ (V_{p+q})_{\lambda,\mu} & \subset & \dots & \subset & (V_n)_{\lambda,\mu} & \subset & (V_{n+1})_{\lambda,\mu} & \subset & \dots & \subset & \bigcup_{n\in\mathbb{Z}_{>0}} (V_n)_{\lambda,\mu} = V_{\lambda,\mu} \end{array} . \tag{7.6}$$

The irreducibility of the $\mathfrak{gl}(n)$-module $(V_n)_{\lambda,\mu}$ for each n implies the irreducibility of the $\mathfrak{gl}(\infty)$-module $V_{\lambda,\mu}$. Similarly, the isomorphism (7.4) follows from the isomorphisms (7.5), and it only remains to show that $V_{\lambda,\mu}$ is a highest weight $\mathfrak{gl}(\infty)$-module with highest weight χ.

For each $n \geq p+q$, the highest weight subspace $(V_n)_{\lambda,\mu}[\chi]$ is one-dimensional, and therefore

$$\begin{aligned} (V_{p+q})_{\lambda,\mu}[\chi] = \cdots = (V_n)_{\lambda,\mu}[\chi] = (V_{n+1})_{\lambda,\mu}[\chi] = \cdots = \bigcup_{n\in\mathbb{Z}_{>0}} (V_n)_{\lambda,\mu}[\chi] \\ = V_{\lambda,\mu}[\chi]. \end{aligned}$$

We also have

$$\mathfrak{b} \cdot V_{\lambda,\mu}[\chi] = \bigcup_{n\in\mathbb{Z}_{>0}} \big(\mathfrak{b}_n \cdot (V_n)_{\lambda,\mu}[\chi]\big) = \bigcup_{n\in\mathbb{Z}_{>0}} (V_n)_{\lambda,\mu}[\chi] = V_{\lambda,\mu}[\chi],$$

which means that any nonzero $v \in V_{\lambda,\mu}[\chi]$ is a highest weight vector generating the $\mathfrak{gl}_\infty$-module $(V_n)_{\lambda,\mu}$. This completes the proof of the theorem. □

The socle filtration of $T^{p,q}$ is described explicitly by the following theorem from [PSty], which shows, in particular, that $T^{\{p,q\}}$ is equal to the socle of $T^{p,q}$.

Theorem 7.4 *Let p, q be nonnegative integers, and let $\ell = \min(p,q)$. Then the Loewy length of the $\mathfrak{gl}(\infty)$-module $T^{p,q}$ equals $\ell+1$, and*

$$\operatorname{soc}^{k-1} T^{p,q} = \bigcap_{I_1,\dots,I_k} \ker\Big(\Phi_{I_1} \circ \cdots \circ \Phi_{I_k} : T^{p,q} \to T^{p-k,q-k}\Big), \quad k = 1, \dots, \ell, \tag{7.7}$$

where the intersection is taken over all k-tuples $I_1, \dots, I_k$ of nonintersecting pairs of indices. Moreover, the socle filtration of $T^{p,q}$, regarded as an $\mathfrak{sl}(\infty)$-module, coincides with (7.7).

Proof Denote by $\mathcal{F}^{(k)}$ the subspaces of $T^{p,q}$ on the right-hand side of (7.7), and for each n set

$$F_n^{(k)} = \bigcap_{I_1,\dots,I_k} \ker\left(\Phi_{I_1} \circ \cdots \circ \Phi_{I_k} : T_n^{p,q} \to T_n^{p-k,q-k}\right), \qquad k = 1, \dots, \ell,$$

where $\{V_n\}_{n\in\mathbb{Z}_{>0}}$ is the exhaustion of V used in the proof of Theorem 7.3. Then we have the commutative diagram of $\mathfrak{gl}(\infty) \times (S_p \times S_q)$-modules

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
F_n^{(1)} & \subset & F_n^{(2)} & \subset & \dots & \subset & F_n^{(\ell)} & \subset & T_n^{p,q} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
F_{n+1}^{(1)} & \subset & F_{n+1}^{(2)} & \subset & \dots & \subset & F_{n+1}^{(\ell)} & \subset & T_{n+1}^{p,q} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathcal{F}^{(1)} & \subset & \mathcal{F}^{(2)} & \subset & \dots & \subset & \mathcal{F}^{(\ell)} & \subset & T^{p,q}
\end{array},$$

in which the vertical arrows are obtained as restrictions of the inclusions in the rightmost column, and yield exhaustions of $\mathcal{F}^{(k)}$. Denote for convenience $\mathcal{F}^{(0)} = 0$, $\mathcal{F}^{(\ell+1)} = T^{p,q}$, and similarly $F_n^{(0)} = 0$, $F_n^{(\ell+1)} = T_n^{p,q}$ for all n. It is easy to check that the induced maps $F_n^{(k+1)}/F_n^{(k)} \to F_{n+1}^{(k+1)}/F_{n+1}^{(k)}$ are injective for all n, r, and therefore for each $r = 0, 1, \dots, \ell$ the layer $\mathcal{F}^{(k+1)}/\mathcal{F}^{(k)}$ is the union of quotients $F_n^{(k+1)}/F_n^{(k)}$.

It is a standard exercise (do it!) to show that for each k, there exists a $\mathfrak{gl}(n) \times (S_p \times S_q)$-module isomorphism

$$F_n^{(k+1)}/F_n^{(k)} \simeq \bigoplus_{|\lambda|=p-k} \bigoplus_{|\mu|=q-k} (V_n)_{\lambda,\mu} \otimes Y(\lambda, \mu; k), \tag{7.8}$$

where $Y(\lambda, \mu; k)$ are $S_p \times S_q$-modules. As in the proof of Theorem 7.3, it follows that

$$\mathcal{F}^{(k+1)}/\mathcal{F}^{(k)} \simeq \bigoplus_{|\lambda|=p-k} \bigoplus_{|\mu|=q-k} V_{\lambda,\mu} \otimes Y(\lambda, \mu; k). \tag{7.9}$$

In particular, this shows that $T^{p,q}$ has a finite Jordan–Hölder series with irreducible constituents of the form $V_{\lambda,\mu}$ for appropriate λ, μ. Moreover, $\mathcal{F}^{(k)}$ is characterized as the unique submodule of $T^{p,q}$, such that for any λ, μ

$$[\mathcal{F}^{(k)} : V_{\lambda,\mu}] = \begin{cases} [T^{p,q} : V_{\lambda,\mu}] & \text{if } |\lambda| > p-k \text{ and } |\mu| > q-k, \\ 0 & \text{otherwise .} \end{cases}$$

We use induction on k to prove: $\mathcal{F}^{(k)} = \operatorname{soc}^{k-1} T^{p,q}$. The base of induction $k = 0$ is trivial since $\mathcal{F}^{(0)} = \operatorname{soc}^{-1} T^{p,q} = \{0\}$.

Suppose that $\mathcal{F}^{(k)} = \operatorname{soc}^{k-1} T^{p,q}$ for some k. Then the quotient $\mathcal{F}^{(k+1)}/\mathcal{F}^{(k)}$ is a semisimple $\mathfrak{gl}(\infty)$-module, implying that $\mathcal{F}^{(k+1)} \subset \operatorname{soc}^{k} T^{p,q}$. Now let W be a simple submodule of $T^{p,q}/\mathcal{F}^{(k)}$. From the description of the filtration $\{\mathcal{F}^{(k)}\}$ given in (7.9), it is clear that $W \simeq V_{\lambda,\mu}$ for some λ, μ satisfying $|\lambda| = p - s$ and $|\mu| = q - s$ with $s \geq k$. In particular, $W \subset \mathcal{F}^{(s+1)}/\mathcal{F}^{(k)}$. Our goal is to show that $s = k$, since this would imply that $\operatorname{soc}^{k} T^{p,q} \subset \mathcal{F}^{(k+1)}$, proving the induction step.

Fix a vector $u \in T^{p,q}$ of weight χ, such that the image of u under the projection $T^{p,q} \to T^{p,q}/\mathcal{F}^{(k)}$ generates the module W. Fix a large enough $m \in \mathbb{Z}_{>0}$ such that $u \in T^{p,q}_m$. We may further assume without loss of generality that u generates a $\mathfrak{gl}(m)$-submodule of $T^{p,q}_m$ isomorphic to $(V_m)_{\lambda,\mu}$.

For each $n > m$, denote by $\pi_n : F^{(s+1)}_n \to (V_n)_{\lambda,\mu} \otimes Y(\lambda, \mu; k)$ the projection corresponding to the decomposition (7.8). The map π_n can be described explicitly, and it is shown in the appendix of [PSty] that $u - \pi_n(u) \notin F^{(s-1)}_n$ for infinitely many n. On the other hand, u generates a submodule of $F^{(s+1)}_n/F^{(k)}_n$ isomorphic to $(V_n)_{\lambda,\mu}$, which implies that $u - \pi_n(u) \in F^{(k)}_n$ for all n. We conclude that $s \leq k$, which implies that in fact $s = k$. The induction is now complete, and thus $\mathcal{F}^{(k)} = \operatorname{soc}^{k-1} T^{p,q}$ for all r.

Finally, we need to verify that the filtration $\{\mathcal{F}^{(k)}\}$ is also the socle filtration of $T^{p,q}$ regarded as an $\mathfrak{sl}(\infty)$-module. The irreducibility of $(V_n)_{\lambda,\mu}$ as an $\mathfrak{sl}(\infty)$-module implies that the layers of the filtration $\mathcal{F}^{(k)}$ remain semisimple. Therefore $\mathcal{F}^{(k)} \subset \operatorname{soc}^{k-1} T^{p,q}$ for all k. The proof of the opposite inclusion, given above for $\mathfrak{gl}(\infty)$, works without any alterations for $\mathfrak{sl}(\infty)$ as well, and this completes the proof of Theorem 7.4. □

We are now ready to prove Theorem 7.2.

Proof of Theorem 7.2 For any partitions $\lambda, \mu, \lambda', \mu'$, we have

$$[(\tilde{V}_n)_{\lambda,\mu} : (V_n)_{\lambda',\mu'}] = \sum_{\gamma} N^{\lambda}_{\lambda',\gamma} N^{\mu}_{\mu',\gamma}$$

provided that n is large enough (see e.g., [HTW]). It follows that

$$[\tilde{V}_{\lambda,\mu} : V_{\lambda',\mu'}] = \sum_{\gamma} N^{\lambda}_{\lambda',\gamma} N^{\mu}_{\mu',\gamma}, \tag{7.10}$$

which when combined with the description of the socle filtration of $T^{p,q}$ yields (7.2). In particular, $\operatorname{soc} \tilde{V}_{\lambda,\mu} \simeq V_{\lambda,\mu}$, and the simplicity of the socle implies the indecomposability of the module $\tilde{V}_{\lambda,\mu}$, concluding the proof. □

In the following examples, we represent the structure of the module $T^{p,q}$ over $\mathfrak{gl}(\infty)$ in the form of a diagram. Each tower represents an indecomposable direct summand of a module, and the vertical arrangement of boxes in each tower

represents the structure of the layers of the socle filtration with the bottom box corresponding to the socle. Note that $\emptyset$ denotes the empty partition.

Example 7.5 Layers of the socle filtration for degree 2:

$$T^{2,0} \sim \boxed{V_{(2),\emptyset}} \oplus \boxed{V_{(1,1),\emptyset}},$$

$$T^{1,1} \sim \begin{array}{|c|}\hline V_{\emptyset,\emptyset} \\ \hline V_{(1),(1)} \\ \hline \end{array},$$

$$T^{0,2} \sim \boxed{V_{\emptyset,(2)}} \oplus \boxed{V_{\emptyset,(1,1)}}.$$

○

Example 7.6 Layers of the socle filtration for degree 3:

$$T^{3,0} \sim \boxed{V_{(3),\emptyset}} \oplus 2\boxed{V_{(2,1),\emptyset}} \oplus \boxed{V_{(1,1,1),\emptyset}},$$

$$T^{2,1} \sim \begin{array}{|c|}\hline V_{(1),\emptyset} \\ \hline V_{(2),(1)} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline V_{(1),\emptyset} \\ \hline V_{(1,1),(1)} \\ \hline \end{array},$$

$$T^{1,2} \sim \begin{array}{|c|}\hline V_{\emptyset,(1)} \\ \hline V_{(1),(2)} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline V_{\emptyset,(1)} \\ \hline V_{(1),(1,1)} \\ \hline \end{array},$$

$$T^{0,3} \sim \boxed{V_{\emptyset,(3)}} \oplus 2\boxed{V_{\emptyset,(2,1)}} \oplus \boxed{V_{\emptyset,(1,1,1)}}.$$

○

The next example shows how the structure of a $\mathfrak{g}$-module $T^{p,q}$ becomes more complicated for higher degrees.

Example 7.7 Degree 10. The module $T^{5,5}$ has 49 mutually non-isomorphic indecomposable direct summands, some of which appear with a multiplicity. We list two of them.

$V_{(1),(1)}$
$V_{(1,1),(2)} \oplus V_{(2),(1,1)} \oplus 2V_{(2),(2)}$
$V_{(2,1),(2,1)} \oplus V_{(2,1),(3)} \oplus 2V_{(3),(2,1)} \oplus V_{(3),(3)}$
$V_{(3,1),(2,2)} \oplus V_{(3,1),(3,1)} \oplus V_{(4),(2,2)} \oplus V_{(4),(3,1)}$
$V_{(4,1),(3,2)}$

$V_{\emptyset,\emptyset}$
$V_{(1),(1)}$
$V_{(1,1),(1,1)}$
$V_{(1,1,1),(1,1,1)}$
$V_{(1,1,1,1),(1,1,1,1)}$
$V_{(1,1,1,1,1),(1,1,1,1,1)}$

○

Let's examine the modules T^p for $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$. As mentioned before, the natural representation V for $\mathfrak{g}$ is self-dual, so it suffices to study only tensor products of the form $V^{\otimes p}$, and yet these representations are not semisimple over $\mathfrak{g}$ for any $p \geq 2$. The degree-two case is illustrated in the following example.

Example 7.8 The degree-two module T^2 for $\mathfrak{g} = \mathfrak{o}(\infty)$ decomposes as

$$V \otimes V = S^2(V) \oplus \Lambda^2(V),$$

where $\Lambda^2(V)$ is the adjoint module (and hence simple), and $S^2(V)$ is indecomposable, but not simple. Indeed, as in the finite-dimensional case when $\mathfrak{g} = \mathfrak{o}(n)$, we have $V \simeq V_*$, which yields the exact sequence of $\mathfrak{g}$-modules

$$0 \to V_{(2)} \to S^2(V) \xrightarrow{tr} \mathbb{C} \to 0,$$

where tr is the trace map on $S^2(V) \subset V \otimes V_* \simeq \mathfrak{gl}(\infty)$. However, in contrast to the finite-dimensional case, this sequence does not split for $\mathfrak{g} = \mathfrak{o}(\infty)$, that is the centralizer of $\mathfrak{o}(\infty)$ within $\mathfrak{gl}(\infty)$ is trivial. Indeed, it is well known that the centralizer of $\mathfrak{o}(n)$ inside $\mathfrak{gl}(n)$ is one dimensional and consequently coincides with the center of $\mathfrak{gl}(n)$ (prove it!). Therefore the centralizer of $\mathfrak{o}(\infty)$ within $\mathfrak{gl}(\infty)$ equals the center of $\mathfrak{gl}(\infty)$, and the latter is equal to zero. ◯

We present without proof the result from [PSty] on the $\mathfrak{g}$-module structure of $V^{\otimes p}$ for $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$. Let $\tilde{V}_\lambda$ denote the irreducible $\mathfrak{sl}(\infty)$-module V_λ considered as a $\mathfrak{g}$-module. Then $\tilde{V}_\lambda$ is an indecomposable $\mathfrak{g}$-module, but is not simple in general.

Theorem 7.9 *Let $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{g} = \mathfrak{sp}(\infty)$. Then*

$$V^{\otimes p} \simeq \bigoplus_{|\lambda|=p} \tilde{V}_\lambda \otimes Y_\lambda$$

as $\mathfrak{g} \times S_p$-modules, where for each partition λ, the $\mathfrak{g}$-module $\tilde{V}_\lambda$ is an indecomposable highest weight module with highest weight $\chi = \sum_{i \in \mathbb{Z}_{>0}} \lambda_i \varepsilon_i$ and has a simple socle denoted V_λ.

The socle filtration of $\tilde{V}_\lambda$ has the following layers:

$$\overline{\mathrm{soc}}^k \tilde{V}_\lambda = \bigoplus_{|\gamma|=k} N^\lambda_{2\gamma,\lambda'} V_{\lambda'},$$

for $\mathfrak{g} = \mathfrak{so}(\infty)$, and by

$$\overline{\mathrm{soc}}^k \tilde{V}_\lambda = \bigoplus_{|\gamma|=k} N^\lambda_{(2\gamma)^\perp,\lambda'} V_{\lambda'},$$

for $\mathfrak{g} = \mathfrak{sp}(\infty)$.

Recall that for a partition μ, one defines $\mu^\perp$ to be the conjugate partition obtained from μ by interchanging rows and columns of the Young diagram.

As a corollary of Theorem 7.9, we see that any simple subquotient of $V^{\otimes p}$ is isomorphic to a simple submodule of $V^{\otimes s}$ for some $s \leq p$. Moreover, simple submodules of $V^{\otimes p}$ are naturally parameterized by Young diagrams.

7.2 The Category $\mathbb{T}_{\mathfrak{g}}$

Now that we understand the simple subquotients of the modules $T^{p,q}$, we would like to realize them in a suitable category. In the case that $\mathfrak{g}$ is a finite-dimensional Lie algebra isomorphic to $\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$, the simple objects of the category of finite-dimensional $\mathfrak{g}$-modules are precisely the simple subquotients of $T^{p,q}$. So we would like to introduce a category of $\mathfrak{g}$-modules that, at least for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{sp}(\infty)$, will be a natural analog of the category of finite-dimensional modules over $\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$.

The first obvious candidate for such a category is the category $\mathrm{Int}_{\mathfrak{g}}$ of integrable $\mathfrak{g}$-modules, i.e., $\mathfrak{g}$-modules M with locally finite action for all elements of $\mathfrak{g}$. By definition, an element $x \in \mathfrak{g}$ acts *locally finitely* on M if

$$\dim(\mathrm{span}\{m, g \cdot m, g^2 \cdot m, \ldots\}) < \infty$$

for any $m \in M$. This choice of category is natural, since when $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, by a well-known argument any integrable $\mathfrak{g}$-module is isomorphic to a direct sum of simple finite-dimensional modules. Indeed, if $\mathfrak{g}$ is finite-dimensional semisimple, then any cyclic submodule of an integrable $\mathfrak{g}$-module M is finite dimensional and hence semisimple by Weyl's Theorem. This shows that M is a sum of simple finite-dimensional modules, and by the equivalence of conditions SS 1 and SS 2 in Chapter XVII, Section 2 of [La], the module M is also isomorphic to a direct sum of simple finite-dimensional modules.

If $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, then a $\mathfrak{g}$-module M is integrable if and only if when M is restricted to any semisimple finite-dimensional subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, M is isomorphic to a direct sum of simple finite-dimensional $\mathfrak{a}$-modules. In this case, the category $\mathrm{Int}_{\mathfrak{g}}$ turns out to be "too large" for the purposes of studying the subquotients of $T^{p,q}$. For instance, there are many simple modules that are not isomorphic to subquotients of $T^{p,q}$, and in fact the problem of classifying simple modules in $\mathrm{Int}_{\mathfrak{g}}$ is not tangible. Simple bounded integrable weight modules are classified in Theorem 8.26. For a discussion of some interesting subcategories of $\mathrm{Int}_{\mathfrak{g}}$, see Sect. 8.4.1.

In this section, we will present a subcategory $\mathbb{T}_{\mathfrak{g}}$ of $\mathrm{Int}_{\mathfrak{g}}$ for which the simple subquotients of $T^{p,q}$ are precisely the simple objects of the category and for which the modules $T^{p,q}$ are injective, despite not being injective in $\mathrm{Int}_{\mathfrak{g}}$. This category $\mathbb{T}_{\mathfrak{g}}$ is called the category of tensor modules.

We denote by T the tensor algebra $T(V \oplus V_*)$ for $\mathfrak{g} = \mathfrak{sl}(\infty),\ \mathfrak{gl}(\infty)$ and $T(V)$ for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. The tensor algebra T has a natural grading and filtration:

$$T := \bigoplus_{p\geq 0,\, q\geq 0} T^{p,q} \quad \text{and} \quad T^{\leq r} := \bigoplus_{0\leq p+q\leq r} T^{p,q} \qquad \text{for } \mathfrak{g} = \mathfrak{sl}(\infty),\ \mathfrak{gl}(\infty)$$

and

$$T := \bigoplus_{p\geq 0} T^{p} \quad \text{and} \quad T^{\leq r} := \bigoplus_{0\leq p\leq r} T^{p} \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty).$$

We call a $\mathfrak{g}$-module a *tensor module* if it is isomorphic to a subquotient of a finite direct sum of copies of $T^{\leq r}$ for some r.

The first question to ask is: what properties do tensor modules have in common with finite-dimensional modules over finite-dimensional simple Lie algebras? This question leads to the following definition.

A $\mathfrak{g}$-module M is said to be an *absolute weight module* if M is an $\mathfrak{h}$-weight module for every splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (recall Definition 6.10).

It is clear that if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, then any finite-dimensional $\mathfrak{g}$-module or, more generally a direct sum of finite-dimensional $\mathfrak{g}$-modules is an absolute weight module. An interesting, but little known fact, is that the converse also holds, that is: every absolute weight $\mathfrak{g}$-module is a direct sum of finite-dimensional $\mathfrak{g}$-modules. According to our discussion above of integrable $\mathfrak{g}$-modules, this claim is a corollary of the following proposition.

Proposition 7.10 *Any absolute weight module over a root-reductive Lie algebra* $\mathfrak{g}$ *is integrable.*

Proof A $\mathfrak{g}$-module M is integrable if every element g of $\mathfrak{g}$ acts locally finitely on M. In Theorem 8.1 below, we prove that the elements of $\mathfrak{g}$ that act locally finitely on M form a Lie subalgebra, denoted $\mathfrak{g}[M]$, and called the Fernando–Kac subalgebra associated to M.

If M is an absolute weight module, then any element of a splitting Cartan subalgebra acts locally finitely on M. Now since every semisimple element of $\mathfrak{g}$ lies in some splitting Cartan subalgebra, we conclude that all semisimple elements of g act locally finitely on M. Finally, since the semisimple elements of $\mathfrak{g}$ generate $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{g}[M]$; in other words, all elements of $\mathfrak{g}$ act locally finitely on M. □

In Chap. 6, we discussed Cartan subalgebras of the Lie algebras $\mathfrak{g} = \mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, but did not raise the question of conjugacy of splitting Cartan subalgebras. This question is relevant for Theorem 7.13 below, and we address this now.

Recall that by Chevalley's Theorem, any two Cartan subalgebras of a finite-dimensional Lie algebra $\mathfrak{g}$ are conjugate by an inner automorphism, that is they are conjugate under the action of the adjoint group G. So it is natural to ask whether this result has an analog.

We consider the exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ to be fixed, where $\mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n)$, $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$, $\mathfrak{o}(\infty) = \varinjlim \mathfrak{o}(n)$, $\mathfrak{sp}(\infty) = \varinjlim \mathfrak{sp}(2n)$, and each inclusion is chosen as a "left upper corner" inclusion. We can define the ind-group $G := \varinjlim G_n$, where G_n denotes the adjoint group of $\mathfrak{g}_n$. The following example shows that two splitting Cartan subalgebras of an infinite-dimensional root-reductive Lie algebra $\mathfrak{g}$ are not necessarily conjugate under the action of G.

Example 7.11 Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ and consider the Cartan subalgebra $\mathfrak{h} := \varinjlim \mathfrak{h}_{2n}$ defined inductively as follows. For $\mathfrak{g}_2 = \mathfrak{sl}(2)$, we take the semisimple element $x_1 := \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$. In $\mathfrak{g}_4$ we let

$$x_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and then extend x_1, x_2 to a Cartan subalgebra $\mathfrak{h}_4$ for $\mathfrak{g}_4 = \mathfrak{sl}(4)$.

Suppose that we have chosen a Cartan subalgebra $\mathfrak{h}_{2n} \subset \mathfrak{g}_{2n} = \mathfrak{sl}(2n)$. Let x_{2n} be the block $(n+2) \times (n+2)$ matrix

$$x_{2n} := \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix},$$

where $A \in \mathfrak{sl}(2)$ is the matrix defining x_1. Then x_{2n} is semisimple and in the centralizer of $\mathfrak{g}_n$, so we can extend (not uniquely) the set $\mathfrak{h}_{2n} \cup \{x_{2n}\}$ to a Cartan subalgebra $\mathfrak{h}_{2n+2} \subset \mathfrak{g}_{2n+1}$.

The Cartan subalgebra $\mathfrak{h} := \varinjlim \mathfrak{h}_{2n}$ of $\mathfrak{g}$ is splitting but cannot be conjugated to the diagonal Cartan subalgebra $\mathfrak{h}'$ using the adjoint group G. Indeed, $G = \varinjlim G_n$ and each $g \in G_n$ fixes pointwise $\mathfrak{g}'_n := [C_{\mathfrak{g}}(\mathfrak{g}_n), C_{\mathfrak{g}}(\mathfrak{g}_n)]$, the commutator subalgebra of the centralizer of $\mathfrak{g}_n$ in $\mathfrak{g}$. By construction, $\mathfrak{h} \cap \mathfrak{g}'_n \neq \mathfrak{h}' \cap \mathfrak{g}'_n$ for every $n \in \mathbb{Z}_{>0}$. Hence $\mathfrak{h}$ and $\mathfrak{h}'$ are not conjugate by the action of G. ◯

We define $\tilde{G}$ to be the subgroup of $\operatorname{Aut} V$ consisting of the automorphisms for which the induced automorphism of V^* restricts to an automorphism of V_*. Then clearly $G \subset \tilde{G} \subset \operatorname{Aut} \mathfrak{g}$, and moreover $\tilde{G} = \operatorname{Aut} \mathfrak{g}$ for $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$.

The following result claims that Chevalley's Theorem remains "almost true" for $\mathfrak{g} = \mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ if one replaces the ind-group G by the group $\tilde{G}$ [NP, DPSn].

Proposition 7.12 *All splitting Cartan subalgebras of* $\mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$ *and* $\mathfrak{sp}(\infty)$ *are* $\tilde{G}$*-conjugate, while there are two* $\tilde{G}$*-conjugacy classes for* $\mathfrak{o}(\infty)$*, corresponding to the two exhaustions* $\varinjlim \mathfrak{o}(2n)$ *and* $\varinjlim \mathfrak{o}(2n+1)$.

For each $\gamma \in \operatorname{Aut} \mathfrak{g}$ and $\mathfrak{g}$-module M, we define the $\mathfrak{g}$-module M^{γ} to be the module with the underlying vector space M and $\mathfrak{g}$-module structure "twisted by

γ", i.e., $x \cdot m := \gamma(x) \cdot m$ for $m \in M^\gamma$ and $x \in \mathfrak{g}$. A $\mathfrak{g}$-module M is called $\tilde{G}$-*invariant*, if for any $\gamma \in \tilde{G}$ there is a $\mathfrak{g}$-module isomorphism $M^\gamma \simeq M$. For $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, a Lie subalgebra of $\mathfrak{g}$ is said to be of *finite corank* if it contains the commutator subalgebra of the centralizer of some finite-dimensional Lie subalgebra of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{gl}(\infty)$, we need the following slightly stronger definition: a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is of *finite corank* if there exist finite-dimensional subspaces $W \subset V$ and $W_* \subset V_*$ such that the restriction of the canonical pairing to $W \otimes W_* \to \mathbb{C}$ is non-degenerate and $\mathfrak{k} \supset \mathfrak{g} \cap (W_*^\perp \otimes W^\perp)$. One reason this latter definition is stronger is that if we take $\mathfrak{gl}(n) = W \otimes W_*$, then $W_*^\perp \otimes W^\perp$ is isomorphic to $\mathfrak{gl}(\infty)$ and is larger than the commutator subalgebra of the centralizer of $\mathfrak{gl}(n)$.

We have the following characterization of tensor modules from [DPS].

Theorem 7.13 *Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. The following conditions on a $\mathfrak{g}$-module M of finite length are equivalent:*

(1) M is an absolute weight module.
(2) M is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and M is $\tilde{G}$-invariant.
(3) M is integrable and $\mathrm{Ann}_{\mathfrak{g}}\, m$ is finite corank for each $m \in M$.
(4) M is a tensor module.

Proof We will prove the equivalence of *(1)*–*(3)*, but first we show that *(4)* implies *(1)*. The proof of the converse statement, *(1)* implies *(4)* can be found in [DPS].

A tensor module is by definition a subquotient of a direct sum of finitely many copies of $T^{\leq r}$ for some r. The $\mathfrak{g}$-module $T^{\leq r}$ is obviously an absolute weight module, and any subquotient of an absolute weight module is an absolute weight module. This shows that *(4)* implies *(1)*. (Note that the condition that M be of finite length is redundant if *(4)* is satisfied, as any tensor module has finite length.)

Let us show that *(1)* implies *(3)*. We already proved in Proposition 7.10 that a $\mathfrak{g}$-module M satisfying *(1)* is integrable. To prove that $\mathrm{Ann}_{\mathfrak{g}}\, m$ is finite corank for all $m \in M$, consider a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}_n$ is a Cartan subalgebra of $\mathfrak{g}_n$, and let $\mathfrak{b} = \mathfrak{h} \dotplus \mathfrak{n}$ be a Dynkin Borel subalgebra of $\mathfrak{g}$. Fix standard bases e_i, h_i, f_i for the $\mathfrak{sl}(2)$-subalgebras corresponding to simple roots α_i, where $i \in \mathbb{Z}_{>0}$, and let $m \in M$ be a nonzero $\mathfrak{h}$-weight vector.

Choose a set of commuting simple roots $\{\alpha_i \mid i \in I\}$ of $\mathfrak{b}$, where I is some subset of $\mathbb{Z}_{>0}$. The set of semisimple elements $\{h_i + e_i \mid i \in I\}$ is $\tilde{G}$-conjugate to the set $\{h_i \mid i \in I\}$, and can thus be extended to a splitting Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$. Since M is an absolute weight module, there exists a nonzero $\mathfrak{h}'$-weight vector $m' \in M$ such that $m \in U(\mathfrak{g}_n) \cdot m'$ for some n. For almost all i, the elements h_i and e_i commute with $\mathfrak{g}_n$, in which case m is an eigenvector for $h_i + e_i$. Thus $e_i \cdot m$ is a scalar multiple of m. Since M is integrable, e_i acts locally nilpotently, and we conclude that $e_i \cdot m = 0$ for all but finitely many i. By considering the set $\{h_i + f_i \mid i \in I\}$ in place of $\{h_i + e_i \mid i \in I\}$, we see that $f_i \cdot m = 0$ for all but finitely many i, and hence $e_i \cdot m = f_i \cdot m = 0$ for all but finitely many $i \in I$.

We now consider separately each of the possible choices of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}(\infty)$, we may assume that the simple roots of $\mathfrak{b}$ are of the form $\{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$. We first choose the set of commuting simple roots to be $\{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$ and obtain in this way that $e_i \cdot m = f_i \cdot m = 0$ for almost all odd indices i. Next, by choosing the set of commuting simple roots as $\{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$, we have $e_i \cdot m = f_i \cdot m = 0$ for almost all even indices i, hence for almost all i. Therefore, $\mathrm{Ann}_{\mathfrak{g}} m$ is a finite corank subalgebra of $\mathfrak{g} = \mathfrak{sl}(\infty)$. This argument also works for $\mathfrak{g} = \mathfrak{gl}(\infty)$.

For $\mathfrak{g} = \mathfrak{o}(\infty)$, we may assume that the set of simple roots of $\mathfrak{g}$ is $\{-\epsilon_1 - \epsilon_2\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$, and one considers in addition to the two sets of positive roots for $\mathfrak{sl}(\infty)$ the sets $\{-\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$ and $\{-\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$. For $\mathfrak{g} = \mathfrak{sp}(\infty)$, the set of simple roots can be chosen as $\{-2\epsilon_1\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$, and one considers in addition to the two sets of positive roots for $\mathfrak{sl}(\infty)$ the sets $\{-2\epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$ and $\{-2\epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$. The remainder of the argument is then the same as for $\mathfrak{sl}(\infty)$.

Next we prove that *(3)* implies *(2)*.

We first show that a $\mathfrak{g}$-module M satisfying *(3)* is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Fix a finite set $\{m_1, \ldots, m_s\}$ of generators of M. Let $\mathfrak{g}'_n$ be the commutator subalgebra of the centralizer in $\mathfrak{g}$ of $\mathfrak{g}_n$. There exists a finite corank subalgebra that annihilates $m_1, \ldots, m_s$, and hence $\mathfrak{g}'_n$ annihilates $m_1, \ldots, m_s$ for some n. Let $\mathfrak{h}'_n$ be a splitting Cartan subalgebra of $\mathfrak{g}'_n$. Obviously M is semisimple over $\mathfrak{h}'_n$. One can find k and a Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{g}_k$ such that $\mathfrak{h} = \mathfrak{h}'_n + \mathfrak{h}_k$ is a splitting Cartan subalgebra of $\mathfrak{g}$. (If $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$ one can choose $k = n$; if $\mathfrak{g} = \mathfrak{sl}(\infty)$, one can set $k = n + 1$). Since M is integrable, M is semisimple over $\mathfrak{h}_k$. Hence M is semisimple over $\mathfrak{h}$.

To finish the proof that *(3)* implies *(2)*, we need to show that M is $\tilde{G}$-invariant. We define a $\tilde{G}$-module structure on M as follows. Fix $m \in M$. There exists an n such that the vector m is fixed by

$$\tilde{G}'_n := \{\gamma' \in \tilde{G} \mid \gamma'(x) = x \text{ for all } x \in \mathfrak{g}_n\}.$$

Then one has $\tilde{G} = G \cdot \tilde{G}'_n$ (prove it!). Fix $\gamma \in \tilde{G}$. We choose a decomposition $\gamma = \gamma''\gamma'$ so that $\gamma' \in \tilde{G}'_n$ and $\gamma'' \in G$. We set $\gamma(m) := \gamma''(m)$, and note that the action of G on M is well defined because M is assumed to be integrable. This yields a well-defined $\tilde{G}$-module structure on M, since for any other decomposition $\gamma = \bar{\gamma}''\bar{\gamma}'$ as above, one has

$$(\bar{\gamma}'')^{-1}\gamma'' = \bar{\gamma}'(\gamma')^{-1} \in \tilde{G}'_n \cap G = \{\gamma \in G \mid \gamma(x) = x \text{ for all } x \in \mathfrak{g}_n\},$$

which must preserve m.

Now fix $\gamma \in \tilde{G}$ and consider the linear operator

$$\varphi_\gamma : M^\gamma \to M, \quad m \mapsto \gamma^{-1}(m).$$

We claim that φ_γ is a $\mathfrak{g}$-module isomorphism. For this we need to check that $x \cdot \varphi_\gamma(m) = \varphi_\gamma(\gamma(x) \cdot m)$ for any $x \in \mathfrak{g}$ and $m \in M$. We have $x \cdot \varphi_\gamma(m) = x \cdot (\gamma^{-1}(m))) = \varphi_\gamma(\gamma(x \cdot \gamma^{-1}(m)))$, hence it suffices to check that $\gamma(x \cdot \gamma^{-1}(m)) = \gamma(x) \cdot m$ for every $x \in \mathfrak{g}$ and $m \in M$. After choosing a decomposition $\gamma = \gamma''\gamma'$ such that $\gamma'' \in G$ and γ' fixes m, x and $x \cdot m$, all that remains to check is that

$$\gamma''(x \cdot (\gamma'')^{-1}(m)) = \gamma''(x) \cdot m$$

for all $x \in \mathfrak{g}$. This latter equality is the well-known relation between the G-module structure on M and the adjoint action of G on $\mathfrak{g}$.

Finally, we show *(2)* implies *(1)*. It is clear that *(2)* implies a slightly weaker statement, namely that M is a weight module for any splitting Cartan subalgebra belonging to the same $\tilde{G}$-conjugacy class as the given splitting Cartan subalgebra $\mathfrak{h}$. For $\mathfrak{g} = \mathfrak{gl}(\infty), \mathfrak{sl}(\infty), \mathfrak{sp}(\infty)$, this proves *(1)*, as all splitting Cartan subalgebras are conjugate under $\tilde{G}$.

Now consider the case $\mathfrak{g} = \mathfrak{o}(\infty)$, which has two $\tilde{G}$-conjugacy classes of splitting Cartan subalgebras. Note that if M is semisimple over every Cartan subalgebra from one $\tilde{G}$-conjugacy class, then *(3)* holds as follows from the proof of the implication *(1)* $\Rightarrow$ *(3)*. Furthermore, the proof that a $\mathfrak{g}$-module of finite length M satisfying *(3)* is a weight module for some splitting Cartan subalgebra involves a choice of $\mathfrak{g}_n$. For $\mathfrak{g} = \mathfrak{o}(\infty)$, there are two different possible choices, namely $\mathfrak{g}_n = \mathfrak{o}(2n)$ and $\mathfrak{g}_n = \mathfrak{o}(2n+1)$, which in turn produce splitting Cartan subalgebras from the two $\tilde{G}$-conjugacy classes. This shows that in each $\tilde{G}$-conjugacy class there is a splitting Cartan subalgebra of $\mathfrak{g}$ for which M is a weight module, and hence we may conclude that *(2)* implies *(1)*. □

The *category of tensor modules* $\mathbb{T}_\mathfrak{g}$ is defined to be the full subcategory of $\mathfrak{g}$-mod consisting of finite-length modules satisfying the equivalent conditions of Theorem 7.13. The category $\mathbb{T}_\mathfrak{g}$ is an abelian monoidal category with respect to the usual tensor product of $\mathfrak{g}$-modules. In addition, for $\mathfrak{g} = \mathfrak{sl}(\infty)$, the category $\mathbb{T}_\mathfrak{g}$ has an involution $(\cdot)_* : \mathbb{T}_\mathfrak{g} \to \mathbb{T}_\mathfrak{g}$, which one can think of as a "restricted dual." Indeed, in this case any automorphism $w \in (\operatorname{Aut}\mathfrak{g}) \setminus \tilde{G}$ induces the autoequivalence of categories

$$\begin{aligned} w_\mathfrak{g} : \mathbb{T}_\mathfrak{g} &\to \mathbb{T}_\mathfrak{g} \\ M &\mapsto M^w. \end{aligned}$$

However since any object of $\mathbb{T}_\mathfrak{g}$ is $\tilde{G}$-invariant, the functor $w_\mathfrak{g}$ does not depend on the choice of w, and is an involution of $\mathbb{T}_\mathfrak{g}$, i.e., $w_\mathfrak{g}^2 = \mathrm{id}$.

We have seen that the simple objects of $\mathbb{T}_\mathfrak{g}$ are precisely the simple tensor modules. Next we would like to understand the injective objects in $\mathbb{T}_\mathfrak{g}$.

7.3 Injective Objects in $\mathbb{T}_{\mathfrak{g}}$

In [DPS], it was shown that the modules $T^{p,q}$ are injective in the category $\mathbb{T}_{\mathfrak{g}}$, and injective hulls of all simple modules in $\mathbb{T}_{\mathfrak{g}}$ were described. Here we present a more general approach to these results which was developed in [CP2].

Recall that in category theory, there is the very useful definition of a Grothendieck category, see [Po]. For our purposes, we adapt this definition in the following more restrictive setting.

A *(finite) ordered Grothendieck category C of* $\mathfrak{g}$*-modules* is by definition a full subcategory of $\mathfrak{g}$-mod that is closed under taking subquotients and arbitrary direct sums and is equipped with the following structure. The category C has a distinguished collection of objects $X_i \in C$ that is indexed by a partially ordered set $(I, \leq)$ with the property that every element dominates finitely many others (i.e., $\{j \in I \mid j \leq i\}$ is finite), and C has a finite collection of morphisms Θ_i for each $i \in I$ from X_i into various objects X_j for $j < i$, subject to the following conditions:

(G1) every object of C is a sum of subquotients of finite direct sums of X_i;
(G2) $Y_i := (\bigcap_{f \in \Theta_i} \ker f)$ has finite length, and $Y_i = \operatorname{soc} X_i$; write $Y_i = \bigoplus_{S \in \mathcal{S}_i} S^{\oplus p(S)}$, where $\mathcal{S}_i$ is the set of isomorphism classes of distinct simple direct summands of Y_i, and $p(S)$ is the multiplicity of S in Y_i;
(G3) the sets $\mathcal{S}_i$ are disjoint;
(G4) X_i decomposes as $\bigoplus_{S \in \mathcal{S}_i} \widetilde{S}^{\oplus p(S)}$ for modules $\widetilde{S}$ with socle S.

Condition (G2), together with the down-finiteness of I, ensures that all X_i have finite length, and hence so do subquotients of finite direct sums of X_i. In particular, the Jordan–Hölder Theorem holds for such subquotients. In addition, condition (G1) implies that any simple object of C belongs to $\mathcal{S}_i$ for some $i \in I$.

In order to see that C is a Grothendieck category according to the definition from [Po], set $\bar{X}_i := \oplus_j F_i^j$ where $F_i^1 \subset F_i^2 \subset \cdots \subset F_i^{N_i} = X_i$ is the socle filtration of X_i. Then one can check that $X := \oplus_{i \in I} \bar{X}_i$ is a generator for C, that means, every object M of C admits an epimorphism $\tilde{X} \to M$, where $\tilde{X}$ is a direct sum of copies of X over a sufficiently large index set depending on the object M.

We would like to understand the indecomposable injective objects of an ordered Grothendieck category C. The definition of an ordered Grothendieck category allows for a quick characterization of indecomposable injective objects. Recall that a module $Q \in C$ is *injective* if any exact sequence $0 \to Q \to M \to N \to 0$ in the category C splits. An injective hull of a module (defined up to isomorphism) is a smallest injective module containing it. An extension M of a module Q (i.e., $Q \subset M$) is called *essential* if for any submodule $K \subset M$ we have $K \cap Q = \{0\}$ implies $K = \{0\}$. Hence, a module with no proper essential extension is injective, and a maximal essential extension of a module is an injective hull.

The following is a result from [CP2].

Theorem 7.14 *The objects X_i are injective in $\mathcal{C}$. The indecomposable injective objects in $\mathcal{C}$ are isomorphic to arbitrary indecomposable direct summands of the objects X_i.*

Theorem 7.14 is a corollary of the following proposition from [CP2].

Proposition 7.15 *If $\mathcal{C}$ is an ordered Grothendieck category, then for any $i \in I$ and any simple $S \in \mathcal{S}_i$, the object $\widetilde{S}$ is an injective hull of S in $\mathcal{C}$.*

Proof Since the socle of $\widetilde{S}$ is the simple module S, it suffices to show that any subquotient J of a finite direct sum of X_i that is an essential extension of S, in fact embeds into $\widetilde{S}$. The following lemma handles a particular case of this situation, and we will use it to prove the general case.

Lemma 7.16 *Suppose $S \subset J$ is an essential extension. If J is a subquotient of a direct sum $X_i^{\oplus q}$ for some i and some q, then J is isomorphic to a submodule of $\widetilde{S}$.*

Proof Let $\pi : X_i^{\oplus q} \to Z$ be an epimorphism such that $J \subset Z$. We can regard S as a submodule of $\operatorname{soc} Z$ using of the monomorphisms $S \subset J \subset Z$. Moreover, there is a decomposition $\operatorname{soc} Z = T \oplus S$, and since the extension $S \subset J$ is essential, T intersects J trivially. Hence after factoring out by T, we may assume that Z has the simple socle S.

The socle $Y_i^{\oplus q}$ of $X_i^{\oplus q}$ can be decomposed as $U \oplus S$ such that $\pi|_{Y_i^{\oplus q}}$ is the projection onto S. Given condition (G4) above, this decomposition can be lifted to a decomposition of $X_i^{\oplus q}$ as $\widetilde{U} \oplus \widetilde{S}$. The socle U of $\widetilde{U}$ is already in the kernel of π, and all other simple constituents of $\widetilde{U}$ belong to the set $\bigcup_{j<i} \mathcal{S}_j$ which does not contain S. It follows that $\pi(\widetilde{U})$ intersects S trivially, and hence also J trivially. Consequently, J admits a monomorphism into $Z/\pi(\widetilde{U})$. But then $Z/\pi(\widetilde{U})$ is isomorphic to $\widetilde{S}$, since it is a quotient of $\widetilde{S}$ and the restriction of π to the simple socle S of $\widetilde{S}$ is a monomorphism. Hence the claim follows. □

Now for the general case, suppose $S \subset J$ is an essential extension. Let $\varphi : X \to J$ be an epimorphism for some submodule $X \subset \bigoplus_{j \in I} X_j^{\oplus q_j}$ of a finite direct sum. Any simple submodule T of X belonging to $\mathcal{S}_j$ for $j \neq i$ will automatically lie in the kernel of φ (by condition (G3) above). So we may assume that X is a submodule of $X_i^{\oplus q_i}$, which allows us to apply Lemma 7.16 to complete the proof. □

We are now ready to study injective objects in the category of tensor modules $\mathbb{T}_{\mathfrak{g}}$. The category $\mathbb{T}_{\mathfrak{g}}$ is an ordered Grothendieck category of modules, where the poset I is the set of ordered pairs (p, q) of nonnegative integers, and the order on I is the smallest partial order such that $(p, q) \geq (p-1, q-1)$ for positive integers p and q. The object X_i corresponding to the element $i = (p, q)$ in I is by definition $T^{p,q} = V^{\otimes p} \otimes V_*^{\otimes q}$.

For $i = (p, q)$, we define $\Theta_i = \{\Phi_{r,s}^{p,q} \mid 1 \leq r \leq p,\ 1 \leq s \leq q\}$ to be the set of contractions $\Phi_{r,s}^{p,q} : T^{p,q} \to T^{p-1,q-1}$, as defined in (7.3).

It is now straightforward to check that conditions (G1)–(G4) from the definition of an ordered Grothendieck category hold for the category $\mathbb{T}_{\mathfrak{g}}$ with this choice of

X_i and Θ_i. As a result, Proposition 7.15 allows us to deduce from Theorems 7.2 and 7.9 the following result of [DPS].

Theorem 7.17 *For $\mathfrak{g} = \mathfrak{gl}(\infty)$ and $\mathfrak{g} = \mathfrak{sl}(\infty)$, an injective hull in $\mathbb{T}_{\mathfrak{g}}$ of the simple module $V_{\lambda,\mu}$ is $\widetilde{V}_{\lambda,\mu} := V_\lambda \otimes (V_*)_\mu$, which has $V_{\lambda,\mu}$ as its socle.*

For $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$, an injective hull in $\mathbb{T}_{\mathfrak{g}}$ of the simple module V_λ is $\widetilde{V}_\lambda$, which has V_λ as its socle.

Moreover, the modules $V^{\otimes p} \otimes V_*^{\otimes q}$ are injective in $\mathbb{T}_{\mathfrak{g}}$, and every indecomposable injective object of $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to a direct summand of some $V^{\otimes p} \otimes V_*^{\otimes q}$.

We finish this section by noting that the category $\mathbb{T}_{\mathfrak{g}}$ does not have enough projectives. This can be seen in several ways, for instance by observing that there is a well-defined infinite projective system of surjective morphisms

$$\cdots \to \widetilde{V}_{\lambda_n,\lambda_n} \to \cdots \to \widetilde{V}_{\lambda_2,\lambda_2} \to \widetilde{V}_{\lambda_1,\lambda_1} \to \mathbb{C} \to 0,$$

where λ_n is a Young diagram consisting of a single row (or column) of length n. Therefore the trivial module $\mathbb{C}$ does not have a projective cover in $\mathbb{T}_{\mathfrak{sl}(\infty)}$. The reader will modify this argument for the categories $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$.

7.4 Vanishing Theorem for Ext in $\mathbb{T}_{\mathfrak{g}}$

Here we establish a pattern for the vanishing of Ext in $\mathbb{T}_{\mathfrak{g}}$, which is crucial in the proof of the Koszulity of $\mathbb{T}_{\mathfrak{g}}$. First, we need to introduce some notation.

For any partition μ, we set

$$\mu^+ := \{\text{partitions } \mu' \mid |\mu'| = |\mu| + 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\},$$
$$\mu^- := \{\text{partitions } \mu' \mid |\mu'| = |\mu| - 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\}.$$

The following lemma from [DPS] computes the socle filtration of a simple module from $\mathbb{T}_{\mathfrak{g}}$ tensored with the natural module V, and in particular this shows that the Loewy length of such a tensor product is at most 2.

Lemma 7.18 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ or $\mathfrak{g} = \mathfrak{gl}(\infty)$. For any simple object $V_{\lambda,\mu}$ of $\mathbb{T}_{\mathfrak{g}}$, there is an exact sequence of $\mathfrak{g}$-modules*

$$0 \to V_{(\lambda,\mu)^+} \to V \otimes V_{\lambda,\mu} \to V_{(\lambda,\mu)^-} \to 0,$$

where

$$V_{(\lambda,\mu)^+} := \bigoplus_{\lambda' \in \lambda^+} V_{\lambda',\mu} \quad \text{and} \quad V_{(\lambda,\mu)^-} := \bigoplus_{\mu' \in \mu^-} V_{\lambda,\mu'}.$$

Moreover, $V_{(\lambda,\mu)^+} = \operatorname{soc}(V \otimes V_{\lambda,\mu})$.

Proof The semisimplification of $V \otimes V_{\lambda,\mu}$ is isomorphic to $V_{(\lambda,\mu)^+} \oplus V_{(\lambda,\mu)^-}$ by the classical Pieri rule. To obtain the equality $V_{(\lambda,\mu)^+} = \mathrm{soc}(V \otimes V_{\lambda,\mu})$, observe that

$$\mathrm{soc}(V \otimes V_{\lambda,\mu}) \subset \mathrm{soc}(V \otimes \tilde{V}_{\lambda,\mu}) = \mathrm{soc}(T^{|\lambda|+1,|\mu|}) \cap (V \otimes \tilde{V}_{\lambda,\mu}) = V_{(\lambda,\mu)^+},$$

where the last equality follows from Theorem 7.2. On the other hand we see that $V_{(\lambda,\mu)^+} \subset \mathrm{soc}(V \otimes V_{\lambda,\mu})$, since Theorem 7.2 also implies that $|\kappa|+|\nu| < |\lambda|+|\mu|+1$ for any simple constituent $V_{\kappa,\nu}$ of $V \otimes \tilde{V}_{\lambda,\mu}$ not lying in the socle of $V \otimes V_{\lambda,\mu}$. Consequently, $V_{(\lambda,\mu)^+} = \mathrm{soc}(V \otimes V_{\lambda,\mu})$.

Finally, to see that the quotient $(V \otimes V_{\lambda,\mu})/V_{(\lambda,\mu)^+}$ is semisimple, note that all simple subquotients of $(V \otimes V_{\lambda,\mu})/V_{(\lambda,\mu)^+}$ lie in $\mathrm{soc}^1(T^{|\lambda|+1,|\mu|})$, again by Theorem 7.2. □

The following lemma from [DPS] can be proved in a similar manner.

Lemma 7.19 *Let $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{g} = \mathfrak{sp}(\infty)$. For any simple object V_λ of $\mathbb{T}_\mathfrak{g}$, there is an exact sequence of $\mathfrak{g}$-modules*

$$0 \to V_{\lambda^+} \to V \otimes V_\lambda \to V_{\lambda^-} \to 0,$$

where

$$V_{\lambda^+} := \bigoplus_{\lambda' \in \lambda^+} V_{\lambda'} \quad \text{and} \quad V_{\lambda^-} := \bigoplus_{\lambda' \in \lambda^-} V_{\lambda'}.$$

Moreover, $V_{\lambda^+} = \mathrm{soc}(V \otimes V_\lambda)$.

Now we are ready to prove a vanishing property of Ext in $\mathbb{T}_\mathfrak{g}$ [DPS, Proposition 5.4].

Proposition 7.20 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ or $\mathfrak{g} = \mathfrak{gl}(\infty)$. If $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu}) \neq 0$, then*

$$|\lambda| - |\lambda'| = |\mu| - |\mu'| = i.$$

Proof We will prove the claim by induction on $|\lambda| + |\mu|$. The base of induction is $(\lambda, \mu) = (\emptyset, \emptyset)$, and in this case the claim follows from the fact that the module $V_{\emptyset,\emptyset} = \mathbb{C}$ is injective in $\mathbb{T}_\mathfrak{g}$. Suppose that $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu}) \neq 0$. Theorem 7.17 implies $V_{\emptyset,\mu} = (V_*)_\mu$ is injective for any μ, so we can assume $|\lambda| > 0$ without loss of generality. Then there exists a partition η such that $\lambda \in \eta^+$. By Lemma 7.18, the module $V_{\lambda,\mu}$ is a direct summand of $V_{(\eta,\mu)^+}$, so we have $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{(\eta,\mu)^+}) \neq 0$.

Consider the following short exact sequence from Lemma 7.18:

$$0 \to V_{(\eta,\mu)^+} \to V \otimes V_{\eta,\mu} \to V_{(\eta,\mu)^-} \to 0.$$

The associated long exact sequence implies that either $\mathrm{Ext}^{i}_{\mathbb{T}_{\mathfrak{g}}}(V_{\lambda',\mu'}, V \otimes V_{\eta,\mu}) \neq 0$ or $\mathrm{Ext}^{i-1}_{\mathbb{T}_{\mathfrak{g}}}(V_{\lambda',\mu'}, V_{(\eta,\mu)^-}) \neq 0$. In the latter case, the induction hypothesis implies

$$|\eta| - |\lambda'| = (|\mu| - 1) - |\mu'| = i - 1,$$

in which case the claim follows since $|\eta| = |\lambda| - 1$.

Now assume that $\mathrm{Ext}^{i}_{\mathbb{T}_{\mathfrak{g}}}(V_{\lambda',\mu'}, V \otimes V_{\eta,\mu}) \neq 0$, and let

$$0 \to V_{\eta,\mu} \to M_0 \to M_1 \to \dots$$

be a minimal injective resolution of $V_{\eta,\mu}$ in $\mathbb{T}_{\mathfrak{g}}$. By the induction hypothesis, for any pair of partitions ν_1, ν_2 and any $j \in \mathbb{Z}_{\geq 0}$, the nonvanishing of $\mathrm{Ext}^{j}_{\mathbb{T}_{\mathfrak{g}}}(V_{\nu_1,\nu_2}, V_{\eta,\mu})$ implies

$$|\eta| - |\nu_1| = |\mu| - |\nu_2| = j. \tag{7.11}$$

By the minimality of the injective resolution, it has no nontrivial direct sum decomposition, so now $\tilde{V}_{\nu_1,\nu_2}$ appears as a direct summand of M_j only if (7.11) holds. Hence $M_j = \oplus \tilde{V}_{\nu_1,\nu_2}$ for some set of ν_1, ν_2 such that $|\nu_1| = |\eta| - j$ and $|\nu_2| = |\mu| - j$.

Furthermore, since the functor $V \otimes (\cdot)$ is exact,

$$0 \to V \otimes V_{\eta,\mu} \to V \otimes M_0 \to V \otimes M_1 \to \dots$$

is a resolution of $V \otimes V_{\eta,\mu}$. Moreover this resolution is injective, as the modules $V \otimes M_i$ are direct summands of injective modules of the form $\oplus_i T^{p_i,q_i}$, and are hence themselves injective. Thus, $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda',\mu'}, V \otimes M_i) \neq 0$ implies $|\lambda'| = |\eta| - i + 1$ and $|\mu'| = |\mu| - i$, and again the claim follows from the fact that $|\eta| = |\lambda| - 1$. □

In a similar manner, one obtains the following proposition from [DPS].

Proposition 7.21 *Let* $\mathfrak{g} = \mathfrak{o}(\infty)$ *or* $\mathfrak{g} = \mathfrak{sp}(\infty)$. *If* $\mathrm{Ext}^{i}_{\mathbb{T}_{\mathfrak{g}}}(V_{\lambda'}, V_{\lambda}) \neq 0$, *then*

$$|\lambda| - |\lambda'| = 2i.$$

The converse of Proposition 7.20 does not hold in general as the following example shows.

Example 7.22 Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then

$$\mathrm{Ext}^{3}_{\mathbb{T}_{\mathfrak{g}}}(V_{(1,1),(1,1)^{\perp}}, V_{(4,1),(3,2)^{\perp}}) = 0.$$

This follows from Corollary 7.35 below and Example 7.7. ○

However, a partial converse of Proposition 7.20 does hold for Ext^1.

Example 7.23 Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. If $\lambda' \in \lambda^-$ and $\mu' \in \mu^-$, then

$$\mathrm{Ext}^1_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu}) = 1.$$

This follows from Corollary 7.35 and Theorem 7.2 (prove it!). ◯

Note that the Ext's in the category $\mathbb{T}_\mathfrak{g}$ differ essentially from the Ext's in the category $\mathfrak{g}$-mod. For example it was shown in [PS6] that $\mathrm{Ext}^1_\mathfrak{g}(V_{\lambda',\mu'}, V_{\lambda,\mu})$ is uncountable dimensional whenever nonzero, whereas $\mathrm{Ext}^k_{\mathbb{T}_{\mathfrak{sl}(\infty)}}(V_{\lambda',\mu'}, V_{\lambda,\mu})$ is always finite dimensional, which follows from Corollary 7.35. Here are two examples.

Example 7.24 Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Consider the exact sequence of $\mathfrak{g}$-modules

$$0 \to V \to (V_*)^* \to (V_*)^*/V \to 0.$$

The $\mathfrak{g}$-module $(V_*)^*/V$ is isomorphic to the direct sum of an uncountable number of copies of the trivial module $\mathbb{C}$, and any one-dimensional subspace in $(V_*)^*/V$ determines a nonzero element of $\mathrm{Ext}^1_{\mathfrak{sl}(\infty)}(\mathbb{C}, V)$. On the other hand by Proposition 7.20 we have $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{sl}(\infty)}}(\mathbb{C}, V) = 0$. ◯

Example 7.25 The space $\mathrm{Ext}^1_{\mathfrak{sl}(\infty)}(\mathbb{C}, \mathfrak{sl}(\infty))$ is uncountable dimensional, and each nonzero vector of $\mathrm{Ext}^1_{\mathfrak{sl}(\infty)}(\mathbb{C}, \mathfrak{sl}(\infty))$ corresponds to a one-dimensional trivial quotient of $\mathrm{soc}^1((\mathfrak{sl}(\infty)_*)^*)$ (see [PS6]). In contrast, the nonzero vectors of the one-dimensional space $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{sl}(\infty)}}(\mathbb{C}, \mathfrak{sl}(\infty))$ correspond to the unique one-dimensional quotient of $\mathrm{soc}^1((\mathfrak{sl}(\infty)_*)^*)$ which determines the exact sequence

$$0 \to \mathfrak{sl}(\infty) \to V \otimes V_* \to \mathbb{C} \to 0$$

where the middle term is an absolute weight module. ◯

7.5 Koszulity of the Category $\mathbb{T}_\mathfrak{g}$

Now we outline one of the main results from [DPS] claiming that the category $\mathbb{T}_\mathfrak{g}$ is Koszul in the sense that it is antiequivalent to the category of locally unitary finite-dimensional modules over a certain direct limit of finite-dimensional Koszul algebras.

We start by recalling the definition of a Koszul ring. A $\mathbb{Z}_{\geq 0}$-graded ring A is *Koszul* if A_0 is a semisimple ring and if for any two graded A-modules M and N of pure weight $m, n \in \mathbb{Z}$ respectively, we have

$$\mathrm{ext}^i_A(M, N) = 0 \ \text{ unless } \ i = m - n,$$

where ext^i_A denotes the ext-group in the category of $\mathbb{Z}$-graded A-modules. For a more extensive study of Koszul rings, see [BGS] where several equivalent definitions are given. Note that we reserve the notation Ext^i_A for the ext-group in the category of (non-graded) A-modules.

For each $r \in \mathbb{Z}_{\geq 0}$, we let $\mathbb{T}^r_{\mathfrak{g}}$ denote the full abelian subcategory of $\mathbb{T}_{\mathfrak{g}}$ whose simple objects are submodules of $T^{\leq r}$. Then

$$\mathbb{T}_{\mathfrak{g}} = \varinjlim \mathbb{T}^r_{\mathfrak{g}}.$$

Consider the finite-dimensional algebra

$$\mathcal{A}^r_{\mathfrak{g}} := \mathrm{End}_{\mathfrak{g}}\, T^{\leq r}$$

and the direct limit algebra

$$\mathcal{A}_{\mathfrak{g}} = \varinjlim \mathcal{A}^r_{\mathfrak{g}}.$$

Note that $\mathcal{A}_{\mathfrak{g}}$ does not have a unit, but is a direct limit of unital algebras.

Let $\mathcal{A}^r_{\mathfrak{g}}$-fmod denote the category of unitary finite-dimensional $\mathcal{A}^r_{\mathfrak{g}}$-modules, and let $\mathcal{A}_{\mathfrak{g}}$-fmod denote the category of locally unitary finite-dimensional $\mathcal{A}_{\mathfrak{g}}$-modules.

The following is proved in [DPS].

Proposition 7.26 *The functors* $\mathrm{Hom}_{\mathfrak{g}}(\cdot, T^{\leq r})$ *and* $\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(\cdot, T^{\leq r})$ *are mutually inverse antiequivalences of the categories* $\mathbb{T}^r_{\mathfrak{g}}$ *and* $\mathcal{A}^r_{\mathfrak{g}}$-fmod.

Let's define a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}^r_{\mathfrak{g}}$. For this purpose, we need the following fact. If $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{gl}(\infty)$, then

$$\mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{s,t}) = 0 \ \text{ unless } \ p - s = q - t \in \mathbb{Z}_{\geq 0}.$$

Indeed if $\varphi : T^{p,q} \to T^{s,t}$ is an $\mathfrak{sl}(\infty)$-module homomorphism, then $\varphi(T^{p,q})$ is a quotient of $T^{p,q}$ satisfying $\mathrm{soc}(\varphi(T^{p,q})) \subset \mathrm{soc}(T^{s,t})$. Hence the claim follows from the description of the socle filtration of $T^{p,q}$ given in Theorem 7.2. Similarly, for $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, we have

$$\mathrm{Hom}_{\mathfrak{g}}(T^{p}, T^{s}) = 0 \ \text{ unless } \ p - s \in 2\mathbb{Z}_{\geq 0}.$$

So one can define a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}^r_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{gl}(\infty)$ by setting

$$(\mathcal{A}^r_{\mathfrak{g}})_i = \bigoplus_{p+q \leq r} (\mathcal{A}_{\mathfrak{g}})^{p,q}_i, \ \text{ where } \ (\mathcal{A}_{\mathfrak{g}})^{p,q}_i = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-i,q-i}), \tag{7.12}$$

and a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}^r_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ by setting

$$(\mathcal{A}^r_{\mathfrak{g}})_i = \bigoplus_{p \leq r} (\mathcal{A}_{\mathfrak{g}})^p_i, \quad \text{where } (\mathcal{A}_{\mathfrak{g}})^p_i = \mathrm{Hom}_{\mathfrak{g}}(T^p, T^{p-2i}). \tag{7.13}$$

By $\mathbb{C}[\Gamma]$, we denote the group algebra of a group Γ.

Lemma 7.27 *There are algebra isomorphisms*

$$(\mathcal{A}^r_{\mathfrak{g}})_0 \simeq \bigoplus_{p+q \leq r} \mathbb{C}[S_p \times S_q] \qquad \text{for } \mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{gl}(\infty)$$

and

$$(\mathcal{A}^r_{\mathfrak{g}})_0 \simeq \bigoplus_{p \leq r} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty).$$

Consequently, the algebra $(\mathcal{A}^r_{\mathfrak{g}})_0$ is semisimple.

Proof The claim for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{gl}(\infty)$ can be proved using Theorem 7.2 to show that $\mathrm{End}_{\mathfrak{g}}(T^{p,q}) = \mathrm{End}_{\mathfrak{g}}(\mathrm{soc}(T^{p,q})) = \mathbb{C}[S_p \times S_q]$. The case of $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ is similar. □

The algebra $\mathcal{A}_{\mathfrak{g}} = \varinjlim \mathcal{A}^r_{\mathfrak{g}}$ has a natural grading

$$\mathcal{A}_{\mathfrak{g}} = \bigoplus_{i \geq 0} (\mathcal{A}_{\mathfrak{g}})_i,$$

where

$$(\mathcal{A}_{\mathfrak{g}})_i = \bigoplus_{p,q \geq 0} (\mathcal{A}_{\mathfrak{g}})^{p,q}_i \qquad \text{for } \mathfrak{g} = \mathfrak{gl}(\infty), \mathfrak{sl}(\infty),$$

and

$$(\mathcal{A}_{\mathfrak{g}})_i = \bigoplus_{p \geq 0} (\mathcal{A}_{\mathfrak{g}})^p_i \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty).$$

Then we have

$$(\mathcal{A}_{\mathfrak{g}})_0 \simeq \bigoplus_{p,q \geq 0} \mathbb{C}[S_p \times S_q] \qquad \text{for } \mathfrak{g} = \mathfrak{gl}(\infty), \mathfrak{sl}(\infty),$$

and

$$(\mathcal{A}_{\mathfrak{g}})_0 \simeq \bigoplus_{p \geq 0} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty), \ \mathfrak{sp}(\infty).$$

Now we would like to relate the category $\mathcal{A}_{\mathfrak{g}}$-fmod with the categories $\mathcal{A}^r_{\mathfrak{g}}$-fmod for all $r \geq 0$. For $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{gl}(\infty)$, observe that $(\mathcal{A}_{\mathfrak{g}})_i^{p,q}(\mathcal{A}_{\mathfrak{g}})_j^{s,t} = 0$ unless $p = s - j$ and $q = t - j$. Hence for each $r \geq 0$,

$$\bar{\mathcal{A}}^r_{\mathfrak{g}} := \bigoplus_{p+q>r} \bigoplus_{i\geq 0} (\mathcal{A}_{\mathfrak{g}})_i^{p,q}$$

is a $\mathbb{Z}_{\geq 0}$-graded ideal in $\mathcal{A}_{\mathfrak{g}}$ such that $\mathcal{A}^r_{\mathfrak{g}} \oplus \bar{\mathcal{A}}^r_{\mathfrak{g}} = \mathcal{A}_{\mathfrak{g}}$. For $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$,

$$\bar{\mathcal{A}}^r_{\mathfrak{g}} := \bigoplus_{p>r} \bigoplus_{i\geq 0} (\mathcal{A}_{\mathfrak{g}})_i^{p}.$$

Therefore each unitary $\mathcal{A}^r_{\mathfrak{g}}$-module X admits a canonical $\mathcal{A}_{\mathfrak{g}}$-module structure with $\bar{\mathcal{A}}^r_{\mathfrak{g}} X = 0$, and thus becomes a locally unitary $\mathcal{A}_{\mathfrak{g}}$-module. We have shown that

$$\mathcal{A}_{\mathfrak{g}}\text{-fmod} = \varinjlim \mathcal{A}^r_{\mathfrak{g}}\text{-fmod}.$$

Finally, Proposition 7.26 implies the following result from [DPS].

Corollary 7.28 *The functors* $\mathrm{Hom}_{\mathfrak{g}}(\cdot, T)$ *and* $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}}(\cdot, T)$ *are mutually inverse antiequivalences of the categories* $\mathbb{T}_{\mathfrak{g}}$ *and* $\mathcal{A}_{\mathfrak{g}}$-fmod.

It remains to show that ring $\mathcal{A}^r_{\mathfrak{g}}$ is Koszul. For the proof of this fact, we recall the following definition. Any $\mathcal{A}_{\mathfrak{g}}$-module X has a *radical filtration*

$$\cdots \subset \mathrm{rad}^1 X \subset \mathrm{rad}^0 X = \mathrm{rad}\, X \subset X$$

where the *radical* of X, denoted $\mathrm{rad}\, X$, is the joint kernel of all surjective $\mathcal{A}_{\mathfrak{g}}$-homomorphisms $X \to X'$ with X' simple, and $\mathrm{rad}^i X := \mathrm{rad}(\mathrm{rad}^{i-1} X)$.

Theorem 7.29 *The ring* $\mathcal{A}^r_{\mathfrak{g}}$ *is Koszul.*

Proof The ring $(\mathcal{A}^r_{\mathfrak{g}})_0$ is semisimple by Lemma 7.27. We will show that unless $i = m - n$, one has $\mathrm{ext}^i_{\mathcal{A}_{\mathfrak{g}}}(M, N) = 0$ for any simple pure $\mathcal{A}_{\mathfrak{g}}$-modules M, N of respective weights m, n. Since any $\mathcal{A}^r_{\mathfrak{g}}$-module admits a canonical $\mathcal{A}_{\mathfrak{g}}$-module structure, it follows that $\mathrm{ext}^i_{\mathcal{A}^r_{\mathfrak{g}}}(M, N) = 0$ for any simple pure $\mathcal{A}^r_{\mathfrak{g}}$-modules M, N of respective weights m, n unless $i = m - n$. The analogous statement for arbitrary $\mathcal{A}^r_{\mathfrak{g}}$-modules of pure degree follows, since all such modules are semisimple.

For the proof, we assume that $\mathfrak{g} = \mathfrak{sl}(\infty)$. The other cases are similar. For each pair of partitions λ, μ, we let $X_{\lambda,\mu}$ (respectively, $\tilde{X}_{\lambda,\mu}$) denote the simple $\mathcal{A}_{\mathfrak{g}}$-module that is the image of $V_{\lambda,\mu}$ (respectively, $\tilde{V}_{\lambda,\mu}$) under the antiequivalence of Corollary 7.28. Then $\tilde{X}_{\lambda,\mu}$ is a projective cover of $X_{\lambda,\mu}$, since $\tilde{V}_{\lambda,\mu}$ is an injective hull of $V_{\lambda,\mu}$.

Now Corollary 7.28 and Proposition 7.20 imply that

$$\mathrm{Ext}^i_{\mathcal{A}_\mathfrak{g}}(X_{\lambda,\mu}, X_{\lambda',\mu'}) = \mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu}) = 0$$

unless $|\lambda|-|\lambda'| = |\mu|-|\mu'| = i$. In addition, it follows that for a minimal projective resolution of $X_{\lambda,\mu}$ in $\mathcal{A}^r_\mathfrak{g}$-fmod

$$\cdots \to P^1 \to P^0 \to X_{\lambda,\mu} \to 0 \tag{7.14}$$

we have $P^i \simeq \oplus_{\nu_1,\nu_2} \tilde{X}_{\nu_1,\nu_2}$ for some set of pairs ν_1, ν_2 satisfying $|\nu_1| = |\lambda| - i$ and $|\nu_2| = |\mu| - i$.

On the other hand if X_{ν_1,ν_2} is a simple constituent of $\mathrm{rad}^i\, \tilde{X}_{\lambda,\mu}$, or if under the antiequivalence V_{ν_1,ν_2} is a simple constituent of $\mathrm{soc}^i\, \tilde{V}_{\lambda,\mu}$, then $|\lambda| - |\nu_1| = |\mu| - |\nu_2| = i$. Therefore we see that in the resolution (7.14) the image of $\mathrm{rad}^j\, P^i$ lies in $\mathrm{rad}^{j+1}\, P^{i-1}$.

Now it is clear that we can endow the resolution (7.14) with a $\mathbb{Z}$-grading by setting the degree of $X_{\lambda,\mu}$ to be an arbitrary integer n. Indeed, one should take each simple constituent of P^i as an $(\mathcal{A}_\mathfrak{g})_0$-module that lies in $\mathrm{rad}^j\, P_i$ and not in $\mathrm{rad}^{j+1}\, P_i$ to have degree $n+i+j+1$. This immediately implies that $\mathrm{ext}^i_{\mathcal{A}_\mathfrak{g}}(X_{\lambda,\mu}, X_{\lambda',\mu'}) = 0$ unless $|\lambda| - |\lambda'| = |\mu| - |\mu'| = i$. □

7.6 Koszul Self-duality of the Ring $\mathcal{A}_{\mathfrak{sl}(\infty)}$

Here we study in more detail the algebras $\mathcal{A}^r_\mathfrak{g}$ and $\mathcal{A}_\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$. The corresponding algebras for $\mathfrak{g} = \mathfrak{gl}(\infty)$ are the same. The main result of this investigation is the self-duality of $\mathcal{A}^r_\mathfrak{g}$ and of $\mathcal{A}_\mathfrak{g}$. One corollary of this result is the following formula for extensions between simple $\mathfrak{g}$-modules:

$$\dim \mathrm{Ext}^k(V^{\lambda',\mu'}, V^{\lambda,\mu}) = \sum_{|\gamma|=k} N^\lambda_{\gamma,\lambda'} N^\mu_{\gamma^\perp,\mu'}. \tag{7.15}$$

We begin by examining the structure of $\mathcal{A}^r_\mathfrak{g}$. The Koszulity of $\mathcal{A}^r_\mathfrak{g}$ implies that $\mathcal{A}^r_\mathfrak{g}$ is a quadratic algebra generated by $(\mathcal{A}^r_\mathfrak{g})_0$ and $(\mathcal{A}^r_\mathfrak{g})_1$, that is,

$$\mathcal{A}^r_\mathfrak{g} \simeq T_{(\mathcal{A}^r_\mathfrak{g})_0}((\mathcal{A}^r_\mathfrak{g})_1)/(R^r),$$

where (R^r) is the two-sided ideal generated by some $(\mathcal{A}^r_\mathfrak{g})_0$-bimodule R^r in $(\mathcal{A}^r_\mathfrak{g})_1 \otimes_{(\mathcal{A}^r_\mathfrak{g})_0} (\mathcal{A}^r_\mathfrak{g})_1$ (see [BGS]). This implies that

$$\mathcal{A}_\mathfrak{g} \simeq T_{(\mathcal{A}_\mathfrak{g})_0}((\mathcal{A}_\mathfrak{g})_1)/(R),$$

where $R = \varinjlim R^r$.

Following [DPS], we will describe $(\mathcal{A}_{\mathfrak{g}})_1$ and R. For this purpose, we fix inclusions $S_n \subset S_{n+1}$ such that S_{n+1} acts on the set $\{1, 2, \ldots, n+1\}$ and S_n is the stabilizer of $n+1$.

Recall that $(\mathcal{A}_{\mathfrak{g}})_i^{p,q} = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-i,q-i})$ and $(\mathcal{A}_{\mathfrak{g}})_0^{p,q} = \mathbb{C}[S_p \times S_q]$.

Lemma 7.30 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ as a left module over $(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}$ is generated by the contractions $\Phi_{i,j} : T^{p,q} \to T^{p-1,q-1}$, as defined in (7.3).*

Proof Let $\varphi \in (\mathcal{A}_{\mathfrak{g}})_1^{p,q} = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-1,q-1})$. By Theorem 7.4,

$$\operatorname{soc} T^{p,q} = \bigcap_{i \le p,\ j \le q} \ker \Phi_{i,j}.$$

Moreover, Theorem 7.2 implies that $\operatorname{soc}(T^{p,q}) \subset \ker \varphi$. Define

$$\Phi : T^{p,q} \to \bigoplus_{i \le p,\ j \le q} T^{p-1,q-1}$$

as the direct sum $\bigoplus_{i,j} \Phi_{i,j}$. Then there exists

$$\alpha : \bigoplus_{i \le p,\ j \le q} T^{p-1,q-1} \to T^{p-1,q-1}$$

such that $\varphi = \alpha \circ \Phi$. Since $\alpha = \bigoplus_{i,j} \alpha_{i,j}$ for some $\alpha_{i,j} \in (\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1} = \mathbb{C}[S_{p-1} \times S_{q-1}]$, we have $\varphi = \sum_{i,j} \alpha_{i,j} \Phi_{i,j}$, which concludes the proof. □

As a corollary of Lemma 7.30, we obtain that $(\mathcal{A}_{\mathfrak{g}})_1$ is generated by contractions as an $(\mathcal{A}_{\mathfrak{g}})_0$-module.

Lemma 7.31 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ is isomorphic to $\mathbb{C}[S_p \times S_q]$ as a right $(\mathcal{A}_{\mathfrak{g}})_0^{p,q}$-module, and the structure of a left $(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}$-module is given by left multiplication via the fixed inclusion*

$$(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1} = \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q] = (\mathcal{A}_{\mathfrak{g}})_0^{p,q}.$$

Proof It is clear that all contractions $\Phi_{i,j} \in (\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ can be obtained from $\Phi_{p,q}$ via the right $\mathbb{C}[S_p \times S_q]$-module structure of $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$. Thus by Lemma 7.30, as a $\mathbb{C}[S_p \times S_q]$-bimodule, $(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ is generated by the single contraction $\Phi_{p,q}$. Furthermore, the computation

$$\begin{aligned}
&\sum_{\sigma \in S_p \times S_q} a_\sigma \Phi_{p,q} \circ \sigma(v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^*) \\
&= \sum_{\substack{\sigma = (\sigma_1, \sigma_2) \\ \in S_p \times S_q}} a_\sigma \langle v_{\sigma_1(p)}, v^*_{\sigma_2(q)} \rangle (v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(p-1)} \otimes v^*_{\sigma_2(1)} \otimes \cdots \otimes v^*_{\sigma_2(q-1)}) \\
&= 0
\end{aligned}$$

for all $v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^* \in T^{p,q}$ implies $a_\sigma = 0$ for all $\sigma \in S_p \times S_q$. Hence $(\mathcal{A}_\mathfrak{g})_1^{p,q}$ is a free right $\mathbb{C}[S_p \times S_q]$-module of rank 1. On the other hand for any $\sigma \in S_{p-1} \times S_{q-1}$, we have

$$\sigma \circ \Phi_{p,q} = \Phi_{p,q} \circ \sigma.$$

So the claim follows. □

Corollary 7.32 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then there is an isomorphism of $(\mathbb{C}[S_{p-2} \times S_{q-2}], \mathbb{C}[S_p \times S_q])$-bimodules*

$$(\mathcal{A}_\mathfrak{g})_1^{p-1,q-1} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}} (\mathcal{A}_\mathfrak{g})_1^{p,q} \simeq \mathbb{C}[S_p \times S_q],$$

where the right-hand side is a $(\mathbb{C}[S_{p-2} \times S_{q-2}], \mathbb{C}[S_p \times S_q])$-bimodule via the embeddings

$$\mathbb{C}[S_{p-2} \times S_{q-2}] \subset \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q].$$

Moreover,

$$(\mathcal{A}_\mathfrak{g})_1 \otimes_{(\mathcal{A}_\mathfrak{g})_0} (\mathcal{A}_\mathfrak{g})_1 = \bigoplus_{p,q} ((\mathcal{A}_\mathfrak{g})_1^{p-1,q-1} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}} (\mathcal{A}_\mathfrak{g})_1^{p,q}).$$

Lemma 7.33 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Let $S \simeq S_2 \times S_2$ denote the subgroup of $S_p \times S_q$ generated by $(p, p-1)_l$ and $(q, q-1)_r$, where $(i, j)_l$ and $(i, j)_r$ stand for the transpositions in S_p and S_q, respectively. Then $R = \bigoplus_{p,q} R^{p,q}$, where*

$$R^{p,q} = (triv \boxtimes sgn \oplus sgn \boxtimes triv) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p \times S_q],$$

and $triv$ and sgn denote respectively the trivial and sign representations of S_2.

Proof The statement is equivalent to the equality

$$\begin{aligned} R^{p,q} =& (1 - (p, p-1)_l)(1 + (q, q-1)_r)\mathbb{C}[S_p \times S_q] \oplus \\ & (1 + (p, p-1)_l)(1 - (q, q-1)_r)\mathbb{C}[S_p \times S_q], \end{aligned}$$

We have the obvious relations in $\mathcal{A}_\mathfrak{g}$

$$\begin{aligned} \Phi_{p-1,q-1}\Phi_{p,q} &= \Phi_{p-1,q-1}\Phi_{p,q}(p, p-1)_l(q, q-1)_r, \\ \Phi_{p-1,q-1}\Phi_{p,q}(p, p-1)_l &= \Phi_{p-1,q-1}\Phi_{p,q}(q, q-1)_r. \end{aligned}$$

Therefore $1 - (p, p-1)_l(q, q-1)_r, (p, p-1)_l - (q, q-1)_r \in R^{p,q}$. On the other hand, $(1 + (p, p-1)_l)(1 + (q, q-1)_r)$ and $(1 - (p, p-1)_l)(1 - (q, q-1)_r)$

clearly do not belong to $R^{p,q}$. The latter two group algebra elements generate a right $\mathbb{C}[S_p \times S_q]$-submodule $\bar{R}^{p,q} \subset (\mathcal{A}_\mathfrak{g})_1^{p-1,q-1} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}} (\mathcal{A}_\mathfrak{g})_1^{p,q}$, and we have

$$(\mathcal{A}_\mathfrak{g})_1^{p-1,q-1} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}} (\mathcal{A}_\mathfrak{g})_1^{p,q} = R^{p,q} \oplus \bar{R}^{p,q}.$$

Hence the statement follows. □

In order to proceed, we need to recall the definition of the *Koszul dual* of a quadratic algebra. Let $A = \sum_{i \geq 0} A_i$ be a left finite quadratic algebra, say $A = T_{A_0}(A_1)/(R)$. Following [BGS], we define the *left dual* $A^!$ to be the right finite quadratic algebra given by $A^! = T_{A_0}(A_1^*)/(R^\perp)$ where $A_1^* = \mathrm{Hom}_{A_0}(A_1, A_0)$ and $R^\perp$ is the annihilator of R in $A_1^* \otimes_{A_0} A_1^*$ with respect to the obvious pairing $(A_1^* \otimes_{A_0} A_1^*) \otimes (A_1 \otimes_{A_0} A_1) \to A_0$.

Theorem 7.34 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then $\mathcal{A}_\mathfrak{g}^r$ is Koszul self-dual, i.e., $\mathcal{A}_\mathfrak{g}^r \simeq ((\mathcal{A}_\mathfrak{g}^r)^!)^{\mathrm{opp}}$. Furthermore, $\mathcal{A}_\mathfrak{g} \simeq (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$ where $\mathcal{A}_\mathfrak{g}^! := \varinjlim (\mathcal{A}_\mathfrak{g}^r)^!$.*

Proof By definition,

$$(\mathcal{A}_\mathfrak{g}^r)^! = T_{(\mathcal{A}_\mathfrak{g}^r)_0}((\mathcal{A}_\mathfrak{g}^r)_1^*)/(R^{r\perp}).$$

Now Lemma 7.31 implies an isomorphism of $((A_\mathfrak{g})_0^{p,q}, (A_\mathfrak{g})_0^{p-1,q-1})$-bimodules

$$((A_\mathfrak{g})_1^{p,q})^* \simeq \mathbb{C}[S_p \times S_q].$$

Hence we have an isomorphism of $((\mathcal{A}_\mathfrak{g}^r)^!)_0^{\mathrm{opp}}$-bimodules

$$((\mathcal{A}_\mathfrak{g}^r)^!)_1^{\mathrm{opp}} \simeq (\mathcal{A}_\mathfrak{g}^r)_1.$$

Next, one can check that $R^\perp = \bar{R}$, where $\bar{R} := \oplus \bar{R}^{p,q}$, and the modules $\bar{R}^{p,q}$ were introduced in the proof of Lemma 7.33. Therefore $((\mathcal{A}^r)_\mathfrak{g}^!)^{\mathrm{opp}} \simeq T_{(\mathcal{A}_\mathfrak{g}^r)_0}((\mathcal{A}_\mathfrak{g}^r)_1)/(\bar{R}^r)$.

Now consider the automorphism σ of $\mathbb{C}[S_p \times S_q]$ defined for all p and q by $\sigma(s,t) = sgn(t)(s,t)$ for all $s \in S_p$, $t \in S_q$. Since $(\mathcal{A}_\mathfrak{g})_0 = \bigoplus_{p,q} \mathbb{C}[S_p \times S_q]$, we can extend σ to an automorphism of $T_{(\mathcal{A}_\mathfrak{g})_0}((\mathcal{A}_\mathfrak{g})_1)$ by setting $\sigma(x) = x$ for any $x \in (\mathcal{A}_\mathfrak{g})_1$. Then $\sigma(R^{p,q}) = \bar{R}^{p,q}$. Hence σ induces an isomorphism $\mathcal{A}_\mathfrak{g}^r \simeq ((\mathcal{A}_\mathfrak{g}^r)^!)^{\mathrm{opp}}$, and clearly also an isomorphism $\mathcal{A}_\mathfrak{g} \simeq (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$. □

Corollary 7.35 *Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then $\dim \mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu})$ equals the multiplicity of $V_{\lambda',(\mu')^\perp}$ in $\overline{\mathrm{soc}}^i \tilde{V}_{\lambda,\mu^\perp}$ (see (7.2)). This implies formula (7.15).*

Proof We apply [BGS, Theorem 2.10.1] to $\mathcal{A}_\mathfrak{g}^r$ for sufficiently large r. Then $\mathrm{Ext}_{\mathcal{A}_\mathfrak{g}}((\mathcal{A}_\mathfrak{g})_0, (\mathcal{A}_\mathfrak{g})_0)$ is isomorphic to $(\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$ as a graded algebra. Moreover, the simple $\mathcal{A}_\mathfrak{g}$-module $X_{\lambda',\mu'}$, which is the image of $V_{\lambda',\mu'}$ under the antiequivalence of

Corollary 7.28, is isomorphic to $(\mathcal{A}_\mathfrak{g})_0 \mathbb{Y}_{\lambda',\mu'}$ where $\mathbb{Y}_{\lambda',\mu'}$ is the product of Young projectors corresponding to the partitions λ' and μ'. This follows immediately from the fact $\mathbb{Y}_{\lambda',\mu'}$ is a primitive idempotent in $(\mathcal{A}_\mathfrak{g})_0$ and hence also in $\mathcal{A}_\mathfrak{g}$. The projective cover $\tilde{X}_{\lambda',\mu'}$ of $X_{\lambda',\mu'}$ is isomorphic to $\mathcal{A}_\mathfrak{g}\mathbb{Y}_{\lambda',\mu'}$. Therefore we have

$$\dim \operatorname{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu}) = \dim \operatorname{Ext}^i_{\mathcal{A}_\mathfrak{g}}(X_{\lambda,\mu}, X_{\lambda',\mu'}) = \dim \mathbb{Y}_{\lambda',\mu'}(\mathcal{A}^!_\mathfrak{g})^{\mathrm{opp}}_i \mathbb{Y}_{\lambda,\mu}.$$

By Theorem 7.34,

$$\dim \mathbb{Y}_{\lambda',\mu'}(\mathcal{A}^!_\mathfrak{g})^{\mathrm{opp}}_i \mathbb{Y}_{\lambda,\mu} = \dim \mathbb{Y}_{\lambda',(\mu')^\perp}(\mathcal{A}_\mathfrak{g})_i \mathbb{Y}_{\lambda,\mu^\perp}.$$

Furthermore, $\dim \mathbb{Y}_{\lambda',(\mu')^\perp}(\mathcal{A}_\mathfrak{g})_i \mathbb{Y}_{\lambda,\mu^\perp}$ equals the multiplicity of $X_{\lambda',(\mu')^\perp}$ in $\operatorname{rad}^{i-1} \tilde{X}_{\lambda,\mu^\perp}/\operatorname{rad}^i \tilde{X}_{\lambda,\mu^\perp}$, which coincides with the multiplicity of $V_{\lambda',(\mu')^\perp}$ in $\operatorname{soc}^i(\tilde{V}_{\lambda,\mu^\perp})/\operatorname{soc}^{i-1}(\tilde{V}_{\lambda,\mu^\perp})$. □

Corollary 7.36

(1) The blocks of the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ are parameterized by $\mathbb{Z}$. The simple modules $V_{\lambda',\mu'}$ and $V_{\lambda,\mu}$ belong to the same block $\mathbb{T}_{\mathfrak{sl}(\infty)}(i)$ for $i \in \mathbb{Z}$ if and only if $|\lambda| - |\mu| = |\lambda'| - |\mu'| = i$.

(2) Two blocks $\mathbb{T}_{\mathfrak{sl}(\infty)}(i)$ and $\mathbb{T}_{\mathfrak{sl}(\infty)}(j)$ are equivalent if and only if $i = \pm j$.

Proof

(1) Since $\tilde{V}_{\lambda,\mu}$ is an injective hull of $V_{\lambda,\mu}$, $\operatorname{Ext}^1_{\mathbb{T}_\mathfrak{g}}(V_{\lambda',\mu'}, V_{\lambda,\mu}) \neq 0$ if and only if $V_{\lambda',\mu'}$ is isomorphic to a submodule of $\tilde{V}_{\lambda,\mu}/V_{\lambda,\mu}$. By Theorem 7.2, this occurs if and only if $\lambda' \in (\lambda)^+$ and $\mu' \in (\mu)^+$. Now consider the minimal equivalence relation on pairs of partitions for which (λ', μ') and (λ, μ) are equivalent whenever $\lambda' \in (\lambda)^+$ and $\mu' \in (\mu)^+$. It is a simple exercise to show that then (λ', μ') and (λ, μ) are equivalent if and only if $|\lambda| - |\mu| = |\lambda'| - |\mu'|$. The proves the first assertion.

(2) The equivalence of the blocks $\mathbb{T}_{\mathfrak{sl}(\infty)}(i)$ and $\mathbb{T}_{\mathfrak{sl}(\infty)}(-i)$ is established by the functor $(\cdot)^*$. We will show that $\mathbb{T}_{\mathfrak{sl}(\infty)}(i)$ and $\mathbb{T}_{\mathfrak{sl}(\infty)}(j)$ are not equivalent when $i \neq \pm j$. Suppose without loss of generality that $i > 0$, $j \geq 0$. Then the isomorphism classes of simple injective objects in $\mathbb{T}_{\mathfrak{sl}(\infty)}(i)$ are parameterized by the partitions of i, since $\{V_{\lambda,\emptyset} \mid |\lambda| = i\}$ represents the set of isomorphism classes of simple injective objects in $\mathbb{T}_{\mathfrak{sl}(\infty)}(i)$. Since the sets $\{V_{\lambda,\emptyset} \mid |\lambda| = i\}$ and $\{V_{\lambda,\emptyset} \mid |\lambda| = j\}$ have different cardinalities for $i \neq j$, except in the case $i = 1$, $j = 0$, the assertion follows in other cases. Now each of the blocks $\mathbb{T}_{\mathfrak{sl}(\infty)}(0)$ and $\mathbb{T}_{\mathfrak{sl}(\infty)}(1)$ has a single simple injective module, up to isomorphism. However, V has nontrivial extensions by both $V_{(2),(1)}$ and $V_{(1,1),(1)}$, whereas $\mathbb{C}$ has a nontrivial extension only by $V_{(1),(1)}$. This completes the proof.

□

7.7 Equivalence of the Categories $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$

Now we proceed to study the structure of $\mathcal{A}_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$. We will outline the proof from [DPS] for the equivalence of the categories $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$. A different proof of this fact is given in [S2].

Recall that $(\mathcal{A}_{\mathfrak{g}})_i^p = \mathrm{Hom}_{\mathfrak{g}}(T^p, T^{p-2i})$ and $(\mathcal{A}_{\mathfrak{g}})_0^p = \mathbb{C}[S_p]$. The following lemma can be proved similar to the case $\mathfrak{g} = \mathfrak{sl}(\infty)$ from Lemma 7.30.

Lemma 7.37 *If $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{g} = \mathfrak{sp}(\infty)$, then $(\mathcal{A}_{\mathfrak{g}})_1^p$ as a left module over $(\mathcal{A}_{\mathfrak{g}})_0^{p-2}$ is generated by the contractions*

$$\Psi_{i,j} : T^p \to T^{p-2},$$
$$v_1 \otimes \cdots \otimes v_p \mapsto \langle v_i, v_j \rangle (v_1 \otimes \cdots \hat{v}_i \cdots \otimes \cdots \hat{v}_j \cdots \otimes v_p),$$

where $\langle \cdot, \cdot \rangle$ stands for the symmetric bilinear form on V if $\mathfrak{g} = \mathfrak{o}(\infty)$, and, respectively, for the symplectic bilinear form on V if $\mathfrak{g} = \mathfrak{sp}(\infty)$.

We let $S_{p-2} \subset S_p$ denote the stabilizer of p and $p-1$, and let $S' \subset S_p$ be the subgroup generated by the transposition $(p-1, p)$.

Lemma 7.38 *We have*

$$(\mathcal{A}_{\mathfrak{g}})_1^p \simeq triv \otimes_{\mathbb{C}[S']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = \mathfrak{o}(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}})_1^p \simeq sgn \otimes_{\mathbb{C}[S']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = \mathfrak{sp}(\infty).$$

Moreover, left multiplication by $\mathbb{C}[S_{p-2}]$ is well defined, as S' centralizes S_{p-2}.

Proof Lemma 7.37 implies that the contraction $\Psi_{p-1,p}$ generates $(\mathcal{A}_{\mathfrak{g}})_1^p$ as a right $\mathbb{C}[S_p]$-module. So the claim follows from the relation

$$\Psi_{p-1,p} = \pm \Psi_{p-1,p}(p, p-1),$$

where the sign is $+$ for $\mathfrak{g} = \mathfrak{o}(\infty)$ and $-$ for $\mathfrak{g} = \mathfrak{sp}(\infty)$. □

Corollary 7.39 *Let $\mathfrak{g} = \mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. Then*

$$(\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p \simeq L_{\mathfrak{g}} \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p],$$

where

$$L_{\mathfrak{g}} := \begin{cases} triv & for\ \mathfrak{g} = \mathfrak{o}(\infty) \\ sgn \boxtimes sgn & for\ \mathfrak{g} = \mathfrak{sp}(\infty) \end{cases}$$

and $S \simeq S_2 \times S_2$ *is the subgroup generated by* $(p, p-1)$ *and* $(p-2, p-3)$.

In order to describe R, we let $R = \bigoplus_p R^p$, where

$$R^p \subset (\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p.$$

We will need the following decompositions of right S_4-modules:

$$triv \otimes_{\mathbb{C}[S]} \mathbb{C}[S_4] = Y_{(2,1,1)} \oplus Y_{(2,2)} \oplus Y_{(4)}, \tag{7.16}$$

$$(sgn \boxtimes sgn) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_4] = Y_{(3,1)} \oplus Y_{(2,2)} \oplus Y_{(1,1,1,1)}, \tag{7.17}$$

where Y_λ denotes a simple S_4-module

Lemma 7.40 *Let* $S'' \subset S_p$ *be the subgroup isomorphic to* S_4 *that fixes* $1, 2, \ldots, p-4$. *Then*

$$R^p \simeq Y_{(2,1,1)} \otimes_{\mathbb{C}[S'']} \mathbb{C}[S_p] \qquad for\ \mathfrak{g} = \mathfrak{o}(\infty),$$

and

$$R^p \simeq Y_{(3,1)} \otimes_{\mathbb{C}[S'']} \mathbb{C}[S_p] \qquad for\ \mathfrak{g} = \mathfrak{sp}(\infty).$$

Proof Let us deal with the case of $\mathfrak{o}(\infty)$. We consider the following Young projectors in $\mathbb{C}[S''] \simeq \mathbb{C}[S_4]$:

$$\begin{aligned} \mathbb{Y}_{(2,1,1)} =& (1 + (p-1, p))(1 - (p, p-2) - (p, p-3) - (p-2, p-3) + \\ & (p, p-2, p-3) + (p, p-3, p-2)), \end{aligned}$$

$$\mathbb{Y}_{(2,2)} = (1 + (p, p-1))(1 + (p-2, p-3))(1 - (p-2, p))(1 - (p-1, p-3)),$$

and

$$\mathbb{Y}_{(4)} = \sum_{s \in S''} s.$$

By (7.16) we have

$$R^p \subset (\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p = \mathbb{Y}_{(2,1,1)}\mathbb{C}[S_p] \oplus \mathbb{Y}_{(2,2)}\mathbb{C}[S_p] \oplus \mathbb{Y}_{(4)}\mathbb{C}[S_p].$$

By direct inspection one can check that

$$\begin{aligned}
\Psi_{p-3,p-2}\Psi_{p-1,p}\mathbb{Y}_{(2,1,1)} &= 0,\\
\Psi_{p-3,p-2}\Psi_{p-1,p}\mathbb{Y}_{(2,2)} &= 2\Psi_{p-3,p-2}\Psi_{p-1,p} - 2\Psi_{p-3,p}\Psi_{p-1,p-2},\\
\Psi_{p-3,p-2}\Psi_{p-1,p}\mathbb{Y}_{(4)} &= 4\Psi_{p-3,p-2}\Psi_{p-1,p}.
\end{aligned}$$

The claim follows for $\mathfrak{o}(\infty)$.

The case of $\mathfrak{sp}(\infty)$ is left to the reader. □

Corollary 7.41 $\mathcal{A}_{\mathfrak{sp}(\infty)} \simeq \mathcal{A}_{\mathfrak{o}(\infty)}$.

Proof We use the automorphism σ of $\mathbb{C}[S_p]$ that sends s to $sgn(s)s$. □

Corollary 7.42 *The categories $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$ are equivalent.*

The following is proved in [DPS].

Proposition 7.43 *$\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$ have two inequivalent blocks $\mathbb{T}_{\mathfrak{g}}^{ev}$ and $\mathbb{T}_{\mathfrak{g}}^{odd}$ generated by all V_λ with $|\lambda|$ even and odd, respectively.*

7.8 Categories of Tensor Modules for Lie Superalgebras

In the paper [S2], Serganova studies the categories of tensor modules $\mathbb{T}_{\mathfrak{g}}$ over the classical locally simple Lie superalgebras $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{sp}(\infty)$, $\mathfrak{sq}(\infty)$. A tensor functor $\mathbb{T}_{\mathfrak{o}(\infty)} \to \mathbb{T}_{\mathfrak{sp}(\infty)}$ establishing an equivalence of tensor categories is constructed using the Lie superalgebra $\mathfrak{osp}(\infty|\infty)$. It is shown that the monoidal categories $\mathbb{T}_{\mathfrak{o}(\infty)}$, $\mathbb{T}_{\mathfrak{sp}(\infty)}$, $\mathbb{T}_{\mathfrak{osp}(\infty|\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$ are equivalent. Furthermore, it is proved that $\mathbb{T}_{\mathfrak{sl}(\infty|\infty)}$ is tensor equivalent to $\mathbb{T}_{\mathfrak{sl}(\infty)}$. Here we outline some of these results.

In Sect. 4.1.4, the classical locally simple Lie superalgebras were constructed as direct limits of simple classical Lie superalgebras. Alternatively, the Lie superalgebra $\mathfrak{sl}(\infty|\infty)$ can be realized as follows. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace such that $V_{\bar{0}}$ and $V_{\bar{1}}$ are countable dimensional, and take V_* to be a superspace with the non-degenerate pairing $str : V \otimes V_* \to \mathbb{C}$. Then $V \otimes V_*$ is an associative superalgebra with $\mathbb{Z}_2$-grading

$$(V \otimes V_*)_{\bar{0}} = ((V_{\bar{0}} \otimes (V_{\bar{0}})_*) \oplus (V_{\bar{1}} \otimes (V_{\bar{1}})_*)), \tag{7.18}$$

$$(V \otimes V_*)_{\bar{1}} = ((V_{\bar{0}} \otimes (V_{\bar{1}})_*) \oplus (V_{\bar{1}} \otimes (V_{\bar{0}})_*)). \tag{7.19}$$

The Lie superalgebra corresponding to $V \otimes V_*$ via formula (2.1) is $\mathfrak{gl}(\infty, \infty)$, whose even part is isomorphic to $\mathfrak{gl}(\infty) \oplus \mathfrak{gl}(\infty)$ according to the decomposition in (7.18). Here is an explicit formula for the supercommutator in $\mathfrak{gl}(\infty, \infty)$ of homogeneous pure tensors:

$$[v_i \otimes v_j^*, v_k \otimes v_l^*] = (-1)^{\bar{v}_k \bar{v}_j^*} str(v_k \otimes v_j^*) v_i \otimes v_l^* - (-1)^{\bar{v}_i \bar{v}_j^* + \bar{v}_k \bar{v}_j^* + \bar{v}_k \bar{v}_l^*} str(v_i \otimes v_l^*) v_k \otimes v_j^*,$$

where $\bar{x} \in \{0, 1\}$ denotes the parity of a homogeneous element x. The Lie subsuperalgebra of $\mathfrak{gl}(\infty|\infty)$ defined by $\mathfrak{sl}(\infty|\infty) := \mathrm{Ker}(str)$ is locally simple, and hence simple.

One can realize $\mathfrak{osp}(\infty|\infty)$ as the Lie subsuperalgebra of $\mathfrak{gl}(\infty|\infty)$ consisting of all finite rank operators for which a fixed even non-degenerate symmetric form $\omega : S^2(V) \to \mathbb{C}$ is invariant. The even part is $\mathfrak{osp}(\infty|\infty)_{\bar{0}} = \mathfrak{o}(\infty) \oplus \mathfrak{sp}(\infty)$. The Lie superalgebras $\mathfrak{sp}(\infty)$ or $\mathfrak{sq}(\infty)$ can be constructed in a similar manner (see [S2]).

Now let $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$, $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{sp}(\infty)$ or $\mathfrak{sq}(\infty)$. Then $\mathbb{T}_\mathfrak{g}$ is defined to be the full subcategory of $\mathfrak{g}$-mod consisting of modules M that satisfy the following three conditions:

(1) M is integrable over $\mathfrak{g}_{\bar{0}}$ (and therefore over $\mathfrak{g}$),
(2) the annihilator of every vector in M is a finite corank subsuperalgebra in $\mathfrak{g}$,
(3) M has finite length as a $\mathfrak{g}_{\bar{0}}$-module.

The category $\mathbb{T}_\mathfrak{g}$ is an abelian monoidal category.

7.8.1 *The Case of* $\mathfrak{g} = \mathfrak{gl}(\infty|\infty)$

As in the case of $\mathbb{T}_{\mathfrak{gl}(\infty)}$, the simple objects of $\mathbb{T}_{\mathfrak{gl}(\infty|\infty)}$ are parameterized by a pair of partitions λ, μ. For a partition λ, let $\mathbb{S}_\lambda$ denote the corresponding Schur functor, and set $V_\lambda := \mathbb{S}_\lambda(V^{\otimes n})$ where $n = |\lambda|$. Then $\tilde{V}_{\lambda,\mu} := V_\lambda \otimes (V_*)_\mu$ is injective in $\mathbb{T}_\mathfrak{g}$, and the socle of $\tilde{V}_{\lambda,\mu}$ is a simple module $V_{\lambda,\mu}$. All simple objects of $\mathbb{T}_\mathfrak{g}$ (up to isomorphism) are obtained in this way.

Using the decomposition of $\mathfrak{gl}(\infty|\infty)_{\bar{0}}$ from (7.18), we set $\mathfrak{k} = V_{\bar{0}} \otimes (V_{\bar{0}})_*$ and $\mathfrak{l} = V_1 \otimes (V_1)_*$. The functor $R_\mathfrak{k} : \mathbb{T}_\mathfrak{g} \to \mathbb{T}_\mathfrak{k}$ is defined by setting $R_\mathfrak{k}(M) := M^\mathfrak{l}$, where $M^\mathfrak{l}$ denotes the space of $\mathfrak{l}$-invariants in M. A functor $R_\mathfrak{l} : \mathbb{T}_\mathfrak{g} \to \mathbb{T}_\mathfrak{l}$ is defined similarly by setting $R_\mathfrak{l}(M) := M^\mathfrak{k}$. Then $R_\mathfrak{k}(V_{\lambda,\mu}) = (V_{\bar{0}})_{\lambda,\mu}$ and $R_\mathfrak{l}(V_{\lambda,\mu}) = (V_{\bar{1}})_{\lambda^\perp,\mu^\perp}$. Next, define the functor $S_\mathfrak{k} : \mathbb{T}_{\mathfrak{gl}(\infty)} \to \mathbb{T}_{\mathfrak{gl}(\infty|\infty)}$ by letting

$$S_\mathfrak{k}(M) := I(M) / \left(\bigcap_{\varphi \in \mathrm{Hom}(I(M), T(V \oplus V_*))} \mathrm{Ker}\, \varphi \right),$$

where $I(M)$ is the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M$ with the action of $\mathfrak{l}$ on M taken to be trivial. By construction, $S_{\mathfrak{k}}(M)$ is the maximal quotient of $I(M)$ belonging to $\mathbb{T}_{\mathfrak{gl}(\infty|\infty)}$. Moreover, $S_{\mathfrak{k}}$ is left adjoint to $R_{\mathfrak{k}}$. This leads to the following result from [S2].

Theorem 7.44 *The functors $R_{\mathfrak{k}}$ and $S_{\mathfrak{k}}$ establish an equivalence of monoidal categories $\mathbb{T}_{\mathfrak{gl}(\infty|\infty)}$ and $\mathbb{T}_{\mathfrak{gl}(\infty)}$.*

The composition $R_{\mathfrak{l}} \circ S_{\mathfrak{k}} : \mathbb{T}_{\mathfrak{k}} \to \mathbb{T}_{\mathfrak{l}}$ defines an autoequivalence of the category $\mathbb{T}_{\mathfrak{gl}(\infty)}$ that sends $V_{\lambda,\mu}$ to $V_{\lambda^\perp,\mu^\perp}$.

7.8.2 *The Case of* $\mathfrak{g} = \mathfrak{osp}(\infty|\infty)$

As in the case of $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$, the simple objects of $\mathbb{T}_{\mathfrak{osp}(\infty|\infty)}$ are parameterized by a single partition λ, and will be accordingly denoted by V_λ [S2]. The $\mathfrak{g}$-module $\tilde{V}_\lambda := \mathbb{S}_\lambda(V^{\otimes n})$ is injective in $\mathbb{T}_{\mathfrak{g}}$, and has the simple socle V_λ.

Using the decomposition of the even part $\mathfrak{osp}(\infty|\infty)_{\bar{0}} = \mathfrak{o}(\infty) \oplus \mathfrak{sp}(\infty)$, define functors $R_{\mathfrak{o}} : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{o}(\infty)}$ and $R_{\mathfrak{sp}} : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{sp}(\infty)}$ by

$$R_{\mathfrak{o}}(M) = M^{\mathfrak{sp}(\infty)}, \;\; R_{\mathfrak{sp}}(M) = M^{\mathfrak{o}(\infty)}.$$

Then $R_{\mathfrak{o}}(V_\lambda) = (V_{\bar{0}})_\lambda$ and $R_{\mathfrak{sp}}(V_\lambda) = (V_{\bar{1}})_{\lambda^\perp}$. Now define the functors $S_{\mathfrak{o}} : \mathbb{T}_{\mathfrak{o}(\infty)} \to \mathbb{T}_{\mathfrak{g}}$ and $S_{\mathfrak{sp}} : \mathbb{T}_{\mathfrak{sp}(\infty)} \to \mathbb{T}_{\mathfrak{g}}$ by

$$M \mapsto I(M)/\Big(\bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(I(M),T(V))} \ker\varphi\Big),$$

where $I(M)$ denotes the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M$ with the action of $\mathfrak{sp}(\infty)$ (respectively, $\mathfrak{o}(\infty)$) on M taken to be trivial. By construction, $S_{\mathfrak{o}}(M)$ is the maximal quotient of $I(M)$ belonging to $\mathbb{T}_{\mathfrak{osp}(\infty|\infty)}$. Moreover, $S_{\mathfrak{o}}$ is left adjoint to $R_{\mathfrak{o}}$. This leads to the following result from [S2].

Theorem 7.45 *The functors $S_{\mathfrak{o}}$ and $R_{\mathfrak{o}}$ establish an equivalence of the monoidal categories $\mathbb{T}_{\mathfrak{osp}(\infty|\infty)}$ and $\mathbb{T}_{\mathfrak{o}(\infty)}$. Similarly, the functors $S_{\mathfrak{sp}}$ and $R_{\mathfrak{sp}}$ establish an equivalence of the monoidal categories $\mathbb{T}_{\mathfrak{osp}(\infty|\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$.*

7.9 Further Developments

We would like to mention a few applications and generalizations of the category of tensor modules $\mathbb{T}_{\mathfrak{g}}$. First of all, the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has proved useful in categorifying the Boson–Fermion correspondence in [FPS]. Furthermore, the category $\mathbb{T}_{\mathfrak{gl}(\infty|\infty)}$ is used to study the Deligne categories $\mathrm{Rep}(GL_t)$ in [EHS]. Finally, a

direct generalization of $\mathbb{T}_{\mathfrak{sl}(\infty)}$ with applications to $\mathfrak{gl}(m|n)$ has been introduced in [HoyPS].

Analogous categories of modules have been studied for Mackey Lie algebras in [PS6, CP1, CP2]. Let's discuss this in more detail. The Mackey Lie algebra $\mathfrak{gl}^M(V, V_*)$ is defined to be the Lie algebra of all endomorphisms of a vector space V that preserve a fixed nondegenerate pairing $p : V \otimes V_* \to \mathbb{C}$, i.e.,

$$\mathfrak{gl}^M(V, V_*) = \{\varphi \in \mathrm{End}_{\mathbb{C}}(V) \mid \varphi^*(V_*) \subseteq V_*\}.$$

If V is countable dimensional, then by Mackey's Theorem there is only one such pairing p up to isomorphism, as mentioned in Example 6.17 [Mac]. Otherwise, there are many, so here we restrict to the case that p is a split pairing (i.e., V and V_* admit a pair of dual bases with respect to p), and this is of course unique up to isomorphism for any fixed cardinality of $\dim V$.

The Mackey Lie algebra $\mathfrak{gl}^M(V, V_*)$ contains the Lie subalgebra $V \otimes V_*$ (prove this!), which is isomorphic to $\mathfrak{gl}(\infty)$ when V is countable dimensional. Moreover, the natural and conatural modules V, V_* are modules over $\mathfrak{gl}^M(V, V_*)$. Mackey Lie algebras are not locally finite, unless they are finite dimensional. In the case when V is countable dimensional, we can realize $\mathfrak{gl}^M(V, V_*)$ as the space of the infinite matrices, where each row and each column has at most finitely many nonzero entries.

For countable-dimensional V, the category of $\mathfrak{gl}^M(V, V_*)$-modules generated by V and V_*, i.e., the smallest monoidal Serre subcategory of $\mathfrak{gl}^M(V, V_*)$-mod containing V and V_*, is equivalent to $\mathbb{T}_{\mathfrak{gl}(\infty)}$ [PS8]. If we include the quotient V^*/V as one of the generators (i.e., consider the category generated by V, V_* and V^*/V), then we again obtain a universal Koszul tensor category, but now the simple objects are parameterized by three Young diagrams, and not just two [CP2]. If we allow V to have higher cardinality, say $\aleph_k$, then V, V_*, V^*/V generate a Koszul category where the simple objects are parameterized by $k + 3$ Young diagrams. Moreover, this category has a natural universality property, see [CP1].

Chapter 8
Weight Modules

The theory of weight modules over reductive Lie algebras is a well-established branch of representation theory. For root-reductive Lie algebras, the theory of weight modules is less developed but is currently coming into shape and will certainly be developed further in the coming years. In this chapter, we give an introduction to weight modules of finite-dimensional Lie algebras, and then we present some basic results on weights modules of root-reductive Lie algebras.

8.1 Preliminaries

Let's start with some general facts about the notion of locally finite action. For the moment, let $\mathfrak{g}$ be any Lie algebra, and let $x \in \mathfrak{g}$ be a fixed element. We say that x *acts locally finitely* on a $\mathfrak{g}$-module M if

$$\dim(\text{span}\{m, x \cdot m, x^2 \cdot m, \ldots\}) < \infty \text{ for any } m \in M.$$

This is equivalent to saying that M is locally $\mathfrak{k}$-finite for the one-dimensional Lie algebra $\mathfrak{k} = \text{span}\{x\}$ (see Sect. 6.1.2). It is clear that the set of all $x \in \mathfrak{g}$ that act locally finitely on a given module M is a fundamental invariant of M. The following remarkable theorem is due independently to Kac and Fernando [K2, F].

Theorem 8.1 *If $\mathfrak{g}$ is a locally finite Lie algebra and M is any $\mathfrak{g}$-module, then the subset $\mathfrak{g}[M]$ of all elements $x \in \mathfrak{g}$ that act locally finitely on M is a Lie subalgebra of $\mathfrak{g}$.*

Proof We first assume that $\mathfrak{g}$ is finite dimensional and follow Kac's proof of Lemma 1.2 from [K4]. The key idea is to show that $\text{span}\,\mathfrak{g}[M]$ is a Lie subalgebra of $\mathfrak{g}$, and then to deduce the equality $\mathfrak{g}[M] = \text{span}\,\mathfrak{g}[M]$. For the former claim, it suffices to show that $[x, y] \in \text{span}\,\mathfrak{g}[M]$ for any $x, y \in \mathfrak{g}[M]$.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_8

Recall that for any locally finite linear operator x on M, e^x is a well-defined automorphism of the vector space M. Furthermore, it is easy to check that for any two elements $x, y \in \mathfrak{g}[M]$ and $t \in \mathbb{C}$, the equality

$$e^{t \operatorname{ad} x} y = e^{tx} y e^{-tx} \tag{8.1}$$

holds, where both sides of (8.1) are considered as linear operators on M. Indeed, in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ we have $\operatorname{ad} x = L_x - R_x$, where L_x (respectively, R_x) stands for left (respectively, right) multiplication by x. As the linear operators L_x and R_x commute, we have

$$\begin{aligned} e^{\operatorname{ad} x} y &= \sum_{n=0}^{\infty} \frac{(L_x - R_x)^n}{n!} y = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k} L_x^k R_x^{n-k}}{k!(n-k)!} y \\ &= \sum_{k=0}^{\infty} \frac{L_x^k}{k!} \sum_{l=0}^{\infty} \frac{(-1)^l R_x^l}{l!} y = e^x y e^{-x}. \end{aligned}$$

Finally, since $tx, ty \in \mathfrak{g}[M]$ for any $t \in \mathbb{C}$, we obtain $e^{t \operatorname{ad} x} y = e^{tx} y e^{-tx}$.

Now if $x, y \in \mathfrak{g}[M]$ and $t \in \mathbb{C}$, clearly $e^{tx} y e^{-tx}$ also belongs to $\mathfrak{g}[M]$, and by (8.1) we have $e^{t \operatorname{ad} x} y \in \mathfrak{g}[M]$ as well. Consider the element

$$f(t) = \frac{e^{t \operatorname{ad} x} y - y}{t} \in \operatorname{span} \mathfrak{g}[M].$$

By taking the limit, we get $[x, y] = \lim_{t \to \infty} f(t) \in \operatorname{span} \mathfrak{g}[M]$. Hence $\operatorname{span} \mathfrak{g}[M]$ is a Lie subalgebra of $\mathfrak{g}$.

It remains to show $\mathfrak{g}[M] = \operatorname{span} \mathfrak{g}[M]$, or equivalently, $\operatorname{span} \mathfrak{g}[M] \subseteq \mathfrak{g}[M]$. Let $x_1, \ldots, x_s$ be a basis of $\operatorname{span} \mathfrak{g}[M]$ with $x_1, \ldots, x_s \in \mathfrak{g}[M]$, and let $z = \sum_i c_i x_i$. Then by the Poincaré–Birkhoff–Witt Theorem, for any n, the vector $z^n \in U(\operatorname{span} \mathfrak{g}[M])$ is a sum of monomials of the form $c x_1^{a_1} \ldots x_n^{a_n}$. This shows that $\dim \operatorname{span}\{m, zm, z^2 m, \ldots\}$ is finite for any m, i.e., $\operatorname{span} \mathfrak{g}[M] \subseteq \mathfrak{g}[M]$.

In the general case, $\mathfrak{g}$ is a locally finite Lie algebra, and so we have $\mathfrak{g} = \bigcup_i \mathfrak{g}_i$ where each $\mathfrak{g}_i$ is a finite-dimensional Lie algebra. Thus,

$$\mathfrak{g}[M] = \bigcup_i \{\mathfrak{g}[M] \cap \mathfrak{g}_i\} = \bigcup_i \mathfrak{g}_i[M]$$

is a Lie subalgebra of $\mathfrak{g}$. □

We refer to the Lie subalgebra $\mathfrak{g}[M]$ as the *Fernando–Kac subalgebra* associated to M. Note that M is integrable if and only if $\mathfrak{g} = \mathfrak{g}[M]$, i.e., every element of $\mathfrak{g}$ acts locally finitely on M.

Next we turn the game around: instead of considering all vectors in $\mathfrak{g}$ that act locally finitely on a fixed $\mathfrak{g}$-module M, we consider all vectors in M on which a fixed subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ acts locally finitely. More precisely, we say that $m \in M$ is a $\mathfrak{k}$-*locally finite vector* if

$$\dim(\mathrm{span}\{m, k \cdot m, k^2 \cdot m, \ldots\}) < \infty \text{ for any } k \in \mathfrak{k}.$$

This definition is well coordinated with the definition of a locally $\mathfrak{k}$-finite module given in Sect. 6.1.2; if M is a locally $\mathfrak{k}$-finite module, then every $m \in M$ is a $\mathfrak{k}$-locally finite vector.

A second remarkable fact which is relevant to our topic is the following.

Theorem 8.2 *Let $\mathfrak{g}$ be any Lie algebra and let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra whose adjoint action on $\mathfrak{g}$ is locally finite, i.e., $\mathfrak{k} \subset \mathfrak{g}[\mathfrak{g}]$. Then for any $\mathfrak{g}$-module M, the subspace $M^{(\mathfrak{k})}$ of all $\mathfrak{k}$-locally finite vectors in M is a $\mathfrak{g}$-submodule of M.*

Proof We need to check that for any fixed $m \in M^{(\mathfrak{k})}$ and $x \in \mathfrak{g}$, we have $x \cdot m \in M^{(\mathfrak{k})}$. Let $k \in \mathfrak{k}$. Consider $\mathrm{span}\{x \cdot m, k \cdot x \cdot m, k^2 \cdot x \cdot m, \ldots\}$. Rewrite $k^n \cdot x \cdot m$ as $\sum_{i=0}^{n} x_i \cdot k^i \cdot m$, where $x_i \in \mathrm{span}\{x, [k, x], [k, [k, x]], \ldots\}$. (Prove the existence of x_i by induction!) Then

$$\mathrm{span}\{x \cdot m, k \cdot x \cdot m, \ldots\} = (\mathrm{span}\{x, [k, x], [k, [k, x]], \ldots\}) \cdot \mathrm{span}\{m, k \cdot m, k^2 \cdot m, \ldots\},$$

and since both spans in the right-hand side are finite dimensional by assumption, so is also the left-hand side, that is $\mathrm{span}\{x \cdot m, k \cdot x \cdot m, \ldots\}$ is finite dimensional. □

In what follows we call a $\mathfrak{g}$-module M a $(\mathfrak{g}, \mathfrak{k})$-*module* if $\mathfrak{k} \subset \mathfrak{g}[M]$. In this way, Theorem 8.2 states that $M^{(\mathfrak{k})}$ is a $(\mathfrak{g}, \mathfrak{k})$-submodule of the $\mathfrak{g}$-module M under the condition that $\mathfrak{k} \subset \mathfrak{g}[\mathfrak{g}]$.

Corollary 8.3 *If $\mathfrak{g}$ is locally finite (in particular, $\mathfrak{g}$ can be finite dimensional), then for any Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$, the space $M^{(\mathfrak{k})}$ is automatically a $(\mathfrak{g}, \mathfrak{k})$-submodule of M.*

Proof Any subalgebra $\mathfrak{k}$ of a locally finite Lie algebra $\mathfrak{g}$ acts locally finitely on $\mathfrak{g}$, i.e., $\mathfrak{k} \subset \mathfrak{g}[\mathfrak{g}]$. □

Note that the unique maximal locally $\mathfrak{k}$-finite $\mathfrak{k}$-submodule M' of M (i.e., $M' = \{m \in M \mid \dim(U(\mathfrak{k}) \cdot m) < \infty\}$) may be contained properly in $M^{(\mathfrak{k})}$. Indeed, consider for example $\mathfrak{g} = \mathfrak{sl}(\infty)$ acting on the natural module V and set $\mathfrak{k} = \mathfrak{g}$. Then $M' = \{0\}$ and $M^{(\mathfrak{k})} = V$. In fact, as Example 6.4 shows, in general M' does not have to be a $\mathfrak{g}$-submodule of M (here $\mathfrak{k} = \mathfrak{t}$). However, if $\dim \mathfrak{k} < \infty$, then $M' = M^{(\mathfrak{k})}$ (this claim follows from the Poincaré–Birkhoff–Witt Theorem applied to $\mathfrak{k}$, and is left as an exercise for the reader). In particular, if $\dim \mathfrak{g}[M] < \infty$, then M is a locally-finite $\mathfrak{g}[M]$-module, that is, $\dim(U(\mathfrak{g}[M]) \cdot m) < \infty$ for any $m \in M$.

8.2 Weight Modules for Finite-Dimensional Lie Algebras

Here we study some very special $(\mathfrak{g}, \mathfrak{k})$-modules, namely $(\mathfrak{g}, \mathfrak{h})$-modules where $\mathfrak{h}$ is a nilpotent subalgebra of a finite-dimensional Lie algebra $\mathfrak{g}$. Until further notice, $\mathfrak{g}$ will be a finite-dimensional Lie algebra and $\mathfrak{h}$ will be a fixed nonzero nilpotent subalgebra.

8.2.1 The $\mathfrak{h}$-Support of a $(\mathfrak{g}, \mathfrak{h})$-Weight Module

We claim that a $(\mathfrak{g}, \mathfrak{h})$-module M is a direct sum of indecomposable modules which resemble Jordan blocks, i.e., it has the form

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda, \tag{8.2}$$

where

$$M^\lambda := \{m \in M \mid \exists n \in \mathbb{Z}_{\geq 0} : (h - \lambda(h))^n \cdot m = 0, \forall h \in \mathfrak{h}\}.$$

Indeed, any simple finite-dimensional $\mathfrak{h}$-module is 1-dimensional (where $\mathfrak{h}$ acts simply through some element $\lambda \in \mathfrak{h}^*$ such that $\lambda|_{[\mathfrak{h},\mathfrak{h}]} = 0$) and any finite-dimensional $\mathfrak{h}$-module has the form (8.2). If M is a $(\mathfrak{g}, \mathfrak{h})$-module, then any cyclic $\mathfrak{h}$-submodule of M has the form (8.2), and since any sum of $\mathfrak{h}$-modules of this form is again of this form, we can conclude that M itself satisfies (8.2) when considered as an $\mathfrak{h}$-module.

By definition, a $\mathfrak{g}$-module M satisfying (8.2) as an $\mathfrak{h}$-module is called a *generalized $\mathfrak{h}$-weight $\mathfrak{g}$-module*. In this way, a $(\mathfrak{g}, \mathfrak{h})$-module and a generalized $\mathfrak{h}$-weight $\mathfrak{g}$-module are the same in the case that $\mathfrak{h}$ is a nilpotent subalgebra of a finite-dimensional Lie algebra $\mathfrak{g}$.

Finally, an *$\mathfrak{h}$-weight $\mathfrak{g}$-module* M is by definition a generalized $\mathfrak{h}$-weight $\mathfrak{g}$-module that is semisimple as an $\mathfrak{h}$-module. This is equivalent to saying that for any $\lambda \in \mathfrak{h}^*$, we have $M^\lambda = \{m \in M \mid h \cdot m = \lambda(h)m \ \forall h \in \mathfrak{h}\}$, i.e., each M^λ is an *$\mathfrak{h}$-weight space* of M. Later in this chapter we will focus our attention on $\mathfrak{h}$-weight modules, but for now we consider the more general case.

Lemma 8.4 *Let M be simple $\mathfrak{g}$-module and let $x \in \mathfrak{g}^\alpha$ for some nonzero $\alpha \in \mathfrak{h}^*$ (here we consider $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha$ as a generalized $\mathfrak{h}$-weight module). Then $x : M \to M$ is either injective or locally nilpotent.*

Proof The set M'' of all vectors $m \in M$ such that $x^N m = 0$ for some $N \in \mathbb{Z}_{>0}$ is a $\mathfrak{g}$-submodule. If $M'' = \{0\}$, then x acts injectively. Otherwise $M'' = M$ and x acts locally nilpotently. □

Note that in the situation of Lemma 8.4, x acts locally nilpotently on M precisely when x belongs to the Fernando–Kac subalgebra $\mathfrak{g}[M]$.

Let M be a $(\mathfrak{g}, \mathfrak{h})$-module. We define the $\mathfrak{h}$-support of M to be

$$\operatorname{supp}_{\mathfrak{h}} M := \{\lambda \in \mathfrak{h}^* \mid M^{\lambda} \neq 0\}.$$

Our first objective is to give a description of the $\mathfrak{h}$-support of any simple $(\mathfrak{g}, \mathfrak{h})$-module M. Let $\Gamma_M \subset \mathfrak{h}^*$ be the monoid generated by all $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}^{\alpha} \nsubseteq \mathfrak{g}[M]$.

Proposition 8.5 *Let M be a simple $(\mathfrak{g}, \mathfrak{h})$-module. Then we have*

$$\operatorname{supp}_{\mathfrak{h}} M = \operatorname{supp}_{\mathfrak{h}} M_0 + \Gamma_M$$

for some simple finite-dimensional $\mathfrak{g}[M]$-submodule M_0 of M.

Proof Since M is a locally-finite $\mathfrak{g}[M]$-module (i.e., $\dim(U(\mathfrak{g}[M])m) < \infty$ for any $m \in M$), it admits a finite-dimensional simple $\mathfrak{g}[M]$-submodule M_0. Furthermore, as follows from Lemma 8.4, in each generalized weight space $\mathfrak{g}^{\alpha}$ with $\mathfrak{g}^{\alpha} \nsubseteq \mathfrak{g}[M]$, there is a nonzero vector x^{α} that acts freely on M, that is for each m, the set $\{m, x^{\alpha} \cdot m, (x^{\alpha})^2 \cdot m, \ldots\}$ is linearly independent. By iterating the action of all such elements on M_0, we get

$$\operatorname{supp}_{\mathfrak{h}} M_0 + \Gamma_M \subset \operatorname{supp}_{\mathfrak{h}} M.$$

On the other hand, since M is simple there is an obvious surjective homomorphism

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[M])} M_0 \to M.$$

Hence,

$$\operatorname{supp}_{\mathfrak{h}} M \subset \operatorname{supp}_{\mathfrak{h}} U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[M])} M_0.$$

(Check that $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[M])} M_0$ is a $(\mathfrak{g}, \mathfrak{h})$-module so that this latter claim makes sense!) Finally, since $\operatorname{supp}_{\mathfrak{h}} U(\mathfrak{g}) \otimes_{U(\mathfrak{g}[M])} M_0$ equals $\operatorname{supp}_{\mathfrak{h}} M_0 + \Gamma_M$, we obtain the inclusion

$$\operatorname{supp}_{\mathfrak{h}} M \subset \operatorname{supp}_{\mathfrak{h}} M_0 + \Gamma_M,$$

which concludes the proof. □

8.2.2 The Shadow Decomposition

As $\mathfrak{g}$ is a generalized $\mathfrak{h}$-weight module itself ($\mathfrak{h}$ being nilpotent), $\operatorname{supp}_{\mathfrak{h}} \mathfrak{g}$ is a well-defined finite subset of $\mathfrak{h}^*$. By definition, the $\mathfrak{h}$-*roots* $\Delta_{\mathfrak{h}}$ are the set $\operatorname{supp}_{\mathfrak{h}} \mathfrak{g} \backslash \{0\}$. It is possible that $\Delta_{\mathfrak{h}} = \emptyset$: for instance, this is the case when $\mathfrak{g}$ is simple and $\mathfrak{h}$ is the nilpotent radical of a Borel subalgebra. In this case,

$$\mathfrak{g} = \mathfrak{g}^0 = \{x \in \mathfrak{g} \mid \exists n \in \mathbb{Z}_{\geq 0} : (\operatorname{ad} h)^n \cdot x = 0, \forall h \in \mathfrak{h}\}.$$

In what follows, we assume M is a fixed simple $(\mathfrak{g}, \mathfrak{h})$-module. We define a decomposition of the $\mathfrak{h}$-roots $\Delta_{\mathfrak{h}}$ into the 4 disjoint subsets

$$\begin{aligned}
(\Delta_{\mathfrak{h}})_M^F &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \alpha \notin \mathbb{R}_{\geq 0}\Gamma_M, -\alpha \notin \mathbb{R}_{\geq 0}\Gamma_M\},\\
(\Delta_{\mathfrak{h}})_M^I &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \alpha \in \mathbb{R}_{\geq 0}\Gamma_M, -\alpha \in \mathbb{R}_{\geq 0}\Gamma_M\},\\
(\Delta_{\mathfrak{h}})_M^+ &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \alpha \notin \mathbb{R}_{\geq 0}\Gamma_M, -\alpha \in \mathbb{R}_{\geq 0}\Gamma_M\},\\
(\Delta_{\mathfrak{h}})_M^- &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \alpha \in \mathbb{R}_{\geq 0}\Gamma_M, -\alpha \notin \mathbb{R}_{\geq 0}\Gamma_M\}.
\end{aligned}$$

Let l be a $\mathbb{R}$-*ray* in $\mathfrak{h}^*$ generated by some $\alpha \in \Delta_{\mathfrak{h}}$, i.e., $l = \{t\alpha \mid t \in \mathbb{R}_{\geq 0}\}$. We will say that l is M-*infinite* (respectively, M-*finite*), if for every $\kappa \in \operatorname{supp}_{\mathfrak{h}} M$ the set $(\kappa + l) \cap \operatorname{supp}_{\mathfrak{h}} M$ is infinite (respectively, finite).

Proposition 8.6 *If M is a simple $(\mathfrak{g}, \mathfrak{h})$-module, then we have*

$$\begin{aligned}
(\Delta_{\mathfrak{h}})_M^F &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \textit{both } \mathbb{R}_{\geq 0}\alpha \textit{ and } \mathbb{R}_{\leq 0}\alpha \textit{ are } M\textit{-finite}\},\\
(\Delta_{\mathfrak{h}})_M^I &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \textit{both } \mathbb{R}_{\geq 0}\alpha \textit{ and } \mathbb{R}_{\leq 0}\alpha \textit{ are } M\textit{-infinite}\},\\
(\Delta_{\mathfrak{h}})_M^+ &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \mathbb{R}_{\geq 0}\alpha \textit{ is } M\textit{-finite}, \mathbb{R}_{\leq 0}\alpha \textit{ is } M\textit{-infinite}\},\\
(\Delta_{\mathfrak{h}})_M^- &:= \{\alpha \in \Delta_{\mathfrak{h}} \mid \mathbb{R}_{\geq 0}\alpha \textit{ is } M\textit{-infinite}, \mathbb{R}_{\leq 0}\alpha \textit{ is } M\textit{-finite}\}.
\end{aligned}$$

Proof Let $\alpha \in \Delta_{\mathfrak{h}}$ and $l = \mathbb{R}_{\geq 0}\alpha$. It is clear that $l \cap \Gamma_M$ is infinite when $l \cap \Gamma_M \neq \{0\}$. If $l \cap \Gamma_M$ is infinite, then Proposition 8.5 implies that $(\kappa + l) \cap \operatorname{supp}_{\mathfrak{h}} M$ is also infinite for every $\kappa \in \operatorname{supp}_{\mathfrak{h}} M$. Therefore, all it remains to show is that $(\kappa + l) \cap \operatorname{supp}_{\mathfrak{h}} M$ is finite whenever $l \cap \Gamma_M = \{0\}$. As $\operatorname{supp}_{\mathfrak{h}} M = \operatorname{supp}_{\mathfrak{h}} M_0 + \Gamma_M$ and $\operatorname{supp}_{\mathfrak{h}} M_0$ is a finite set, it suffices to restrict to the case $\operatorname{supp}_{\mathfrak{h}} M = \Gamma_M$. The claim then follows from the following general result. □

Lemma 8.7 *Let Γ be a finitely generated submonoid in $\mathbb{R}^n$ and l be a ray in $\mathbb{R}^n$ satisfying $l \cap \Gamma = \{0\}$. Then $(\nu + l) \cap \Gamma$ is finite for any $\nu \in \Gamma$.*

Proof Let X be the subgroup generated by Γ, which must be isomorphic to $\mathbb{Z}^m$ for some m, and let V be the vector space over $\mathbb{Q}$ spanned by X (m might be larger than n). If $\tilde{l}$ is the line containing l, then $\tilde{l} \cap V$ is a vector subspace of V. Moreover, $l \cap V$ is a cone inside $\tilde{l} \cap V$. We now pass to the ambient space $\mathbb{R}^m \cong \mathbb{R} \otimes_{\mathbb{Q}} V$, where the

group X is identified with $\mathbb{Z}^m$, and we set $H = \mathbb{R} \otimes_{\mathbb{Q}} (\tilde{l} \cap V)$, and let H^+ be the closure of the cone $l \cap V$. We have $H^+ \cap \Gamma = \{0\}$, and to finish the proof, it suffices to show that $(\nu + H^+) \cap \Gamma$ is finite for any $\nu \in \Gamma$.

Let C be the convex hull of Γ in $\mathbb{R}^m$. Clearly, C is a piecewise linear cone whose faces lie in rational subspaces. We claim that $H^+ \cap C = \{0\}$. Suppose to the contrary that $H^+ \cap C \neq \{0\}$, and consider the cone $H \cap C$. Its faces are rational in H. Take a face C' of $H \cap C$ that is contained in H^+. It necessarily contains a rational point $\gamma \neq 0$. Since $\gamma \in C \cap \mathbb{Q}^m$, we have $k\gamma \in \Gamma$ for some $k \in \mathbb{Z}_{>0}$. But then $k\gamma \in H^+ \cap \Gamma$, which is a contradiction. This shows that $H^+ \cap C = \{0\}$.

The next step is to show that $(\nu + H^+) \cap C$ is bounded in $\mathbb{R}^m$ for any $\nu \in C$. Indeed, for any $\beta \in H^+$, there is a linear function f on $\mathbb{R}^m$ such that $f(C) \geq 0$ and $f(\beta) < 0$. Therefore $\nu + t\beta \notin C$ for $t > -\frac{f(\nu)}{f(\beta)}$.

Finally, note that $(\nu + H^+) \cap C$ is bounded if and only if $(\nu + H^+) \cap C \cap \mathbb{Z}^m$ is finite. As $(\nu + H^+) \cap C$ is bounded, $(\nu + H^+) \cap \Gamma$ must be finite. □

We now introduce the $\mathfrak{h}$-*shadow decomposition* of $\mathfrak{g}$ corresponding to any simple $(\mathfrak{g}, \mathfrak{h})$-module M:

$$\mathfrak{g} = (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+ \oplus \mathfrak{g}_M^-, \tag{8.3}$$

where

$$\mathfrak{g}_M^F := \mathfrak{h}' \oplus \left(\bigoplus_{\alpha \in (\Delta_{\mathfrak{h}})_M^F} \mathfrak{g}^\alpha \right), \quad \mathfrak{g}_M^I := \mathfrak{h}' \oplus \left(\bigoplus_{\alpha \in (\Delta_{\mathfrak{h}})_M^I} \mathfrak{g}^\alpha \right), \quad \mathfrak{g}_M^\pm := \bigoplus_{\alpha \in (\Delta_{\mathfrak{h}})_M^\pm} \mathfrak{g}^\alpha,$$

with $\mathfrak{h}'$ being by definition the generalized $\mathfrak{h}$-weight space $\mathfrak{g}^0$. The decomposition (8.3) is only a decomposition of vector spaces.

The $\mathfrak{h}$-shadow decomposition depends only on the Fernando–Kac subalgebra $\mathfrak{g}[M]$ and not on the module M. However as we shall see in the next section, the shadow decomposition or even the support $\operatorname{supp}_{\mathfrak{h}} M$ does not determine the Lie subalgebra $\mathfrak{g}[M] \subset \mathfrak{g}$. The term shadow decomposition appears already in [DMP] and [DP1].

The following statement is a corollary of Proposition 8.5 and Lemma 8.7.

Corollary 8.8 *Let M be a simple $(\mathfrak{g}, \mathfrak{h})$-module. Then the subspaces $\mathfrak{g}_M^I$, $\mathfrak{g}_M^+$, $\mathfrak{g}_M^-$ of $\mathfrak{g}$ are Lie subalgebras of $\mathfrak{g}$.*

8.2.3 $(\mathfrak{g}, \mathfrak{h})$-*Modules in a Special Case*

Let $\mathfrak{g} = \mathfrak{r} \dotplus \mathfrak{sl}(2)$, $\mathfrak{r}$ being the (solvable) radical of $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{sl}(2)$.

Fix a standard $\mathfrak{sl}(2)$-basis $\{e, f, h\}$ of $\mathfrak{sl}(2)$ such that $\mathfrak{h} = \mathbb{C}h$. Then $\mathfrak{h}^*$ is naturally identified with $\mathbb{C}$, the roots $\pm\alpha$ of $\mathfrak{sl}(2)$ being represented respectively by $\pm 2 \in \mathbb{C}$. We set

$$\mathfrak{r}^+ := \bigoplus_{t>0} \mathfrak{r}^t, \qquad \mathfrak{r}^- := \bigoplus_{t<0} \mathfrak{r}^t,$$

the $\mathfrak{h}$-weights t being considered as elements of $\mathbb{Z} \subset \mathbb{C} = \mathfrak{h}^*$, and we note that in this case $\mathfrak{h}' = \mathfrak{g}^0$ is nothing but the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, so $\mathfrak{h}' = \mathfrak{r}^0 \supseteq \mathfrak{h}$.

For $\mathfrak{g}$ as above, it is easy to show that any simple generalized $\mathfrak{h}$-weight $\mathfrak{g}$-module M is simply a weight module. (Do it!) Moreover, it is proved in [PS3] that there are the following 7 possibilities for the support of an arbitrary simple weight $\mathfrak{g}$-module:

(1) $\operatorname{supp}_{\mathfrak{h}} M = \{-\lambda, -\lambda+2, \ldots, \lambda-2, \lambda\}$, $\lambda \in \mathbb{Z}$ (finite string);
(2) $\operatorname{supp}_{\mathfrak{h}} M = \{\ldots, \lambda-2, \lambda\}$, $\lambda \in \mathbb{C}$ (reduced negative string);
(2′) $\operatorname{supp}_{\mathfrak{h}} M = \{\ldots, \lambda-2p-3, \lambda-2p-2, \lambda-2p-1, \lambda-2p, \lambda-2p+2, \ldots, \lambda-2, \lambda\}$ $\lambda \in \mathbb{C}$, $p \in \mathbb{Z}_{\geq 0}$;
(3) $\operatorname{supp}_{\mathfrak{h}} M = \{\lambda, \lambda+2, \ldots\}$, $\lambda \in \mathbb{C}$ (reduced positive string);
(3′) $\operatorname{supp}_{\mathfrak{h}} M = \{\lambda, \lambda+2, \ldots, \lambda+2p, \lambda+2p+1, \lambda+2p+2, \lambda+2p+3, \ldots\}$, $\lambda \in \mathbb{C}$, $p \in \mathbb{Z}_{\geq 0}$;
(4) $\operatorname{supp}_{\mathfrak{h}} M = \{\ldots, \lambda-2, \lambda, \lambda+2, \ldots\}$, $\lambda \in \mathbb{C}$;
(4′) $\operatorname{supp}_{\mathfrak{h}} M = \{\ldots, \lambda-2, \lambda-1, \lambda, \lambda+1, \lambda+2, \ldots\}$, $\lambda \in \mathbb{C}$.

The cases (1), (2), (3) and (4) correspond to the case $\mathfrak{r} = 0$ (or, when $\mathfrak{r} \neq 0$, to the case of an irreducible $\mathfrak{h}$-weight module on which $\mathfrak{r}$ acts trivially), and are well known. Case (1) describes an arbitrary simple finite-dimensional $\mathfrak{sl}(2)$-module, cases (2) and (3) correspond to irreducible Verma modules (there are two Borel subalgebras of $\mathfrak{sl}(2)$ containing $\mathfrak{h}$, hence two cases), and case (4) corresponds to non-highest weight simple $(\mathfrak{sl}(2), \mathfrak{h})$-modules, a standard reference for those is [Dix, Exercise 7.8.16]. (Please do this exercise in [Dix] in full detail!)

We urge the reader to spend some time experimenting with the cases (2′), (3′) and (4′) for various solvable radicals $\mathfrak{r}$. We will just give a quick hint about how such examples can be constructed.

To construct an example of a $\mathfrak{g}$-module M as in case (2′) for $p = 1$, let $\mathfrak{r} = L(0) \oplus L(3) \oplus L(4)$ (here $L(i)$ denotes the simple $\mathfrak{sl}(2)$-module of dimension $i+1$). Define the Lie bracket $[\cdot, \cdot] : \Lambda^2\mathfrak{r} \to \mathfrak{r}$ by putting $[L(0) \oplus L(4), \mathfrak{r}] = 0$ and by letting $[\cdot, \cdot] : \Lambda^2 L(3) \to L(0) \oplus L(4)$ be an isomorphism of $\mathfrak{sl}(2)$-modules. As an $\mathfrak{h}$-module, $\mathfrak{g}$ decomposes as $\mathfrak{g} = \oplus_{t=-4}^{4} \mathfrak{g}^t$. Let $\mathfrak{g}^+ = \oplus_{i=0}^{4} \mathfrak{g}^t$ and choose $\lambda \in (\mathfrak{h}')^*$ so that $\lambda(h) \notin \mathbb{Z}_{\geq 0}$, $\lambda([\mathfrak{g}^1, \mathfrak{g}^{-1}]) = 0$ and $\lambda([\mathfrak{g}^3, \mathfrak{g}^{-3}]) \neq 0$. Let c_λ be the 1-dimensional $\mathfrak{h}'$-module defined by λ. Make it a $\mathfrak{g}^+$-module by putting $\mathfrak{g}^t \cdot c_\lambda = 0$ for $t > 0$. Set $\bar{M} := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} c_\lambda$ and let M be the unique simple quotient of $\bar{M}$. Then we claim that $\operatorname{supp}_{\mathfrak{h}} M = \{\ldots, \lambda-4, \lambda-3, \lambda-2, \lambda\}$. Indeed, a simple computation shows that $\mathfrak{g}^1 \cdot \mathfrak{g}^{-1} \cdot c_\lambda = 0$, $\mathfrak{g}^3 \cdot \mathfrak{g}^{-3} \cdot c_\lambda = c_\lambda$. Therefore $M^{\lambda-1} = 0$,

$M^{\lambda-3} \neq 0$. Moreover, the condition $\lambda(h) \notin \mathbb{Z}_{\geq 0}$ implies that f acts freely on M, hence $M^{\lambda-2-n} \neq 0$ for all $n \geq 1$.

To generalize the above example to the case $(2')$ for any integer $p \geq 1$, the reader could choose $\mathfrak{r} = L(2p+1) \oplus \Lambda^2 L(2p+1)$ and then imitate the above construction. An example for the case $(3')$ can be constructed by replacing $\mathfrak{g}^+$ with $\mathfrak{g}^-$.

For the case $(4')$, let $\mathfrak{r}$ be a 3-dimensional Heisenberg algebra with commutation relations $[x, y] = z$, $[x, z] = [y, z] = 0$, and let $\mathfrak{sl}(2)$ act on the span$\{x, y\}$ through the natural representation. (Then $\mathfrak{sl}(2)$ acts trivially on $\mathbb{C}z$). This defines $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{sl}(2)$. Let H be the simple $\mathfrak{r}$-module with basis v_i, $i \in \mathbb{Z}$ such that $x \cdot v_i = v_{i+1}$, $y \cdot v_i = v_{i-1}$, $z \cdot v_i = 0$. Endow H with an $\mathfrak{r} \rtimes \mathfrak{h}$-module structure by putting $h \cdot v_i = (\lambda + 1)v_i$, and consider the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{r} \rtimes \mathfrak{h})} H$. As H is a simple $\mathfrak{h}$-weight module, any simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{r} \rtimes \mathfrak{h})} H$ is an $\mathfrak{h}$-weight $\mathfrak{g}$-module containing H as a $\mathfrak{r} \rtimes \mathfrak{h}$-submodule. The reader will then immediately show that the support of this simple $\mathfrak{h}$-weight $\mathfrak{g}$-module is as in $(4')$.

If M is any simple $\mathfrak{h}$-weight module (with support as in cases (1)–$(4')$, the decomposition (8.3) is simply the following:

$$
\begin{aligned}
(1): &\quad \mathfrak{g} = \mathfrak{g}_M^F;\\
(2), (2'): &\quad \mathfrak{g}_M^F = \mathfrak{g}_M^I = \mathfrak{h}',\ \mathfrak{g}_M^+ = \mathfrak{r}^+ \rtimes \mathbb{C}e,\ \mathfrak{g}_M^- = \mathfrak{r}^- \rtimes \mathbb{C}f;\\
(3), (3'): &\quad \mathfrak{g}_M^F = \mathfrak{g}_M^I = \mathfrak{h}',\ \mathfrak{g}_M^+ = \mathfrak{r}^- \rtimes \mathbb{C}f,\ \mathfrak{g}_M^- = \mathfrak{r}^+ \rtimes \mathbb{C}e;\\
(4), (4'): &\quad \mathfrak{g} = \mathfrak{g}^I.
\end{aligned}
$$

We conclude this example with a list of cases for the Fernando–Kac subalgebra $\mathfrak{g}[M]$:

(1) $\mathfrak{g}[M] = \mathfrak{g}$;
(2) $\mathfrak{g}[M] = (\mathfrak{g}[M] \cap \mathfrak{r}) \rtimes \text{span}\{e, h\}$ and $\mathfrak{g}[M] \cap \mathfrak{r} \supset \mathfrak{r}^+ + (\oplus_{k \in \mathbb{Z}} \mathfrak{r}^{2k+1})$;
$(2')$ $\mathfrak{g}[M] = (\mathfrak{g}[M] \cap \mathfrak{r}) \rtimes \text{span}\{e, h\}$ and $\mathfrak{g}[M] \supset \oplus_{k \geq -2p} \mathfrak{r}^k$, $0 \neq \mathfrak{r}^{-2p-1} \not\subseteq \mathfrak{g}[M]$;
(3) $\mathfrak{g}[M] = (\mathfrak{g}[M] \cap \mathfrak{r}) \rtimes \text{span}\{f, h\}$ and $\mathfrak{g}[M] \supset \mathfrak{r}^- + \oplus_{k \in \mathbb{Z}} \mathfrak{r}^{2k+1}$;
$(3')$ $\mathfrak{g}[M] = (\mathfrak{g}[M] \cap \mathfrak{r}) \rtimes \text{span}\{e, h\}$ and $\mathfrak{g}[M] \supset \oplus_{k \geq -2p} \mathfrak{r}^k$, $0 \neq \mathfrak{r}^{-2p-1} \not\subseteq \mathfrak{g}[M]$;
(4) $\mathfrak{g}[M] = (\mathfrak{g}[M] \cap \mathfrak{r}) \rtimes \mathbb{C}h$ and $\mathfrak{g}[M] \supset \oplus_{k \in \mathbb{Z}} \mathfrak{r}^{2k+1}$;
$(4')$ $\mathfrak{g}[M] = (\mathfrak{g}[M] \cap \mathfrak{r}) \rtimes \mathbb{C}h$ and $\mathfrak{g}[M] \not\subseteq \oplus_{k \in \mathbb{Z}} \mathfrak{r}^{2k+1}$.

Note that the above list is not really explicit, and the reason is that $\text{supp}_{\mathfrak{h}} M$ does not determine $\mathfrak{g}[M]$. As an exercise, the reader should find an appropriate $\mathfrak{r}$ and two simple $\mathfrak{h}$-weight $\mathfrak{r} \rtimes \mathfrak{sl}(2)$-modules M_1, M_2 such that $\text{supp}_{\mathfrak{h}} M_1 = \text{supp}_{\mathfrak{h}} M_2$, but $\mathfrak{g}[M_1] \neq \mathfrak{g}[M_2]$.

8.2.4 *General* $(\mathfrak{g}, \mathfrak{h})$*-Modules*

We now return to the general case, namely, $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$. The results of this section have been established in

[PS1]. For $\alpha \in \Delta_{\mathfrak{h}}$, consider the subalgebra $\mathfrak{g}^{\mathbb{R}\alpha}$ generated by all $\mathfrak{g}^{\beta}$ with $\beta \in \mathbb{R}\alpha$. We call this the *line subalgebra* of $\mathfrak{g}$ corresponding to $\alpha \in \Delta_{\mathfrak{h}}$. It is easy to show (prove this!) that there are two options: a line subalgebra is either solvable, or its Levi component is isomorphic to $\mathfrak{sl}(2)$. In the latter case we call α an $\mathfrak{sl}(2)$-*type root* of $\mathfrak{g}$.

Fix a Levi decomposition $\mathfrak{g} = \mathfrak{r} \leftthreetimes \mathfrak{g}_{ss}$, where $\mathfrak{r}$ denotes the radical of $\mathfrak{g}$ and $\mathfrak{g}_{ss}$ is a Levi component. In order to be able to prove further results about the shadow decomposition (8.3), from now on we assume that $\mathfrak{h}$ contains a Cartan subalgebra $\mathfrak{h}_{ss}$ of $\mathfrak{g}_{ss}$. Under this assumption we have the following key lemma.

Lemma 8.9 *Let M be a simple generalized $\mathfrak{h}$-weight $\mathfrak{g}$-module. Let α, β, $\alpha+\beta$ be $\mathfrak{sl}(2)$-type roots of $\mathfrak{g}$. If $\mathbb{R}_{\geq 0}\alpha$ and $\mathbb{R}_{\geq 0}\beta$ are M-finite, then $\mathbb{R}_{\geq 0}(\alpha+\beta)$ is also M-finite.*

Proof Assume to the contrary that $\mathbb{R}_{\geq 0}(\alpha+\beta)$ is M-infinite. Then we are forced to assume that both rays $\mathbb{R}_{\leq 0}\alpha$ and $\mathbb{R}_{\leq 0}\beta$ are M-finite. Indeed, if $\mathbb{R}_{\leq 0}\alpha$ (respectively, $\mathbb{R}_{\leq 0}\beta$) were infinite, this would imply that $\mathbb{R}_{\geq 0}\alpha$ (respectively, $\mathbb{R}_{\geq 0}\beta$) is infinite by Corollary 8.8.

Consider now the semisimple part $\mathfrak{s}$ of the Lie algebra generated by $\mathfrak{g}^{\mathbb{R}\alpha}$ and $\mathfrak{g}^{\mathbb{R}\beta}$. It is not difficult to see that $\mathfrak{s}$ is a simple Lie algebra of rank 2 (prove this!), and we can assume that $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$. Furthermore, $\mathfrak{s} \subset \mathfrak{g}[M]$ since all four rays $\mathbb{R}_{\geq 0}\alpha$, $\mathbb{R}_{\leq 0}\alpha$, $\mathbb{R}_{\geq 0}\beta$, $\mathbb{R}_{\leq 0}\beta$ are M-finite. Therefore $\operatorname{supp} M$ is $W_{\mathfrak{s}}$-invariant, $W_{\mathfrak{s}}$ being the Weyl group of $\mathfrak{s}$. This yields a contradiction. Indeed, if all roots of $\mathfrak{s}$ have the same length, then there is $w \in W_{\mathfrak{s}}$ such that $w(\alpha) \in \mathbb{R}_{\geq 0}(\alpha+\beta)$. Then $\mathbb{R}_{\geq 0}(\alpha+\beta)$ must be M-finite, a contradiction. If not all roots of $\mathfrak{s}$ have the same length, there is $w \in W_{\mathfrak{s}}$ such that $\mathbb{R}_{\geq 0}\alpha \subset \mathbb{R}_{\geq 0}(\alpha+\beta) + w(\mathbb{R}_{\geq 0}(\alpha+\beta))$, again a contradiction. This shows that $\mathbb{R}_{\geq 0}(\alpha+\beta)$ is M-finite. □

Now let $\mathfrak{g}_0[M]$ denote $\mathfrak{z} \leftthreetimes \mathfrak{g}[M]_{ss}$, where $\mathfrak{z}$ is the centralizer of $\mathfrak{g}_{ss}$ in the subalgebra of $\mathfrak{g}[M]$ generated by $\mathfrak{g}[M]_{ss}$ and $\mathfrak{h}$. Note that any root of $\mathfrak{g}_0[M]$ is an $\mathfrak{sl}(2)$-type root of $\mathfrak{g}$.

Corollary 8.10

(1) If $\alpha \in \Delta_{\mathfrak{h}}$ is a root of $\mathfrak{g}_0[M]$, then $\alpha \in \Delta_M^F$ or $\alpha \in \Delta_M^I$.
(2) If α, β are roots of $\mathfrak{g}_0[M]$ and $\alpha \in \Delta_M^F$, $\beta \in \Delta_M^I$, then

$$[\mathfrak{g}_0[M]^{\alpha}, \mathfrak{g}_0[M]^{\beta}] = 0.$$

(3) Both $\mathfrak{g}_0[M]^F := (\mathfrak{g}_0[M] \cap \mathfrak{g}_M^F)$ and $\mathfrak{g}_0[M]^I := (\mathfrak{g}_0[M] \cap \mathfrak{g}_M^I)$ are Lie subalgebras of $\mathfrak{g}$, and

$$\mathfrak{g}_0[M] = \mathfrak{g}_0[M]^F + \mathfrak{g}_0[M]^I, \quad \mathfrak{g}_0[M]^F \cap \mathfrak{g}_0[M]^I = \mathfrak{h} \cap \mathfrak{g}_0[M].$$

Proof

(1) If $\alpha \in (\Delta_\mathfrak{h})^{\pm}_M$, then $-\alpha \in (\Delta_\mathfrak{h})^{\pm}_M$. On the other hand, by the proof of Lemma 8.9, $\mathbb{R}_{\geq 0}\alpha$ and $\mathbb{R}_{\leq 0}\alpha$ will be both M-finite or M-infinite as $-\alpha = w_\alpha(\alpha)$ for $w_\alpha \in W_\mathfrak{s}$. Hence $\alpha \in (\Delta_\mathfrak{h})^F_M$ or $\alpha \in (\Delta_\mathfrak{h})^I_M$.

(2) If $0 \neq [\mathfrak{g}_0[M^\alpha], \mathfrak{g}_0[M]^\beta] \subset \mathfrak{g}_0[M]^\gamma$, then $\mathbb{R}_{\geq 0}\gamma$ is either M-finite or M-infinite. Both possibilities contradict Lemma 8.9. (Prove this!) Hence $[\mathfrak{g}_0[M^\alpha], \mathfrak{g}_0[M]^\beta] = 0$.

(3) The fact that $(\mathfrak{g}_0[M] \cap \mathfrak{g}^F_M)$ is a Lie subalgebra follows directly from Lemma 8.9. The remaining statements follow from *(1)* and Corollary 8.8.

□

Lemma 8.11 *Let $\bar{\Gamma}_M$ be the submonoid of $\mathfrak{h}^*$ generated by Γ_M and by the roots of $\mathfrak{g}_0[M]^I$. Then $\operatorname{supp}_\mathfrak{h} M$ is $\bar{\Gamma}_M$-invariant.*

Proof Let α be a root of $\mathfrak{g}_0[M]^I$. Then α is an $\mathfrak{sl}(2)$-root, $\mathbb{R}_\alpha \cap \mathbb{Z}_{\geq 0}\Delta_\mathfrak{h}$ equals $\mathbb{Z}_{\geq 0}\alpha$ or $\mathbb{Z}_{\geq 0}\frac{1}{2}\alpha$. By assumption, $\mathbb{R}\alpha \cap \Gamma_M$ is an infinite monoid, therefore $k\alpha \in \Gamma_M$ for some $k \in \mathbb{Z}_{\geq 0}$. In this way, for any $\kappa \in \operatorname{supp}_\mathfrak{h} M$, we have $\kappa + k\alpha \in \operatorname{supp}_\mathfrak{h} M$, and thus also $\kappa + \alpha \in \operatorname{supp}_\mathfrak{h} M$ by standard $\mathfrak{sl}(2)$-theory (prove this latter claim!). □

The following explicit description of the support was proved in [PS1].

Theorem 8.12 *If $\mathfrak{h}$ is a nilpotent subalgebra containing a Cartan subalgebra $\mathfrak{h}_{ss}$ of $\mathfrak{g}_{ss}$ and M is a simple $(\mathfrak{g}, \mathfrak{h})$-module, then*

$$\operatorname{supp}_\mathfrak{h} M = \operatorname{supp}_\mathfrak{h} M^F_0 + \bar{\Gamma}_M \tag{8.4}$$

for some simple (finite-dimensional) $\mathfrak{g}_0[M]^F$-submodule M^F_0 of M.

Proof Let $M_0 \subset M$ be a simple $\mathfrak{g}[M]$-submodule of M as in the proof of Proposition 8.5. Note that M is simple as a $\mathfrak{g}_0[M]$-module since $\mathfrak{g}_0[M]_{ss} = \mathfrak{g}[M]_{ss}$. Fix a simple $\mathfrak{g}_0[M]^F$-submodule M^F_0 of M_0. Then $\operatorname{supp}_\mathfrak{h} M \supset \operatorname{supp}_\mathfrak{h} M^F_0 + \bar{\Gamma}_M$ by Lemma 8.11. On the other hand, $\mathfrak{g}_0[M] = \mathfrak{g}_0[M]^F + \mathfrak{g}_0[M]^I$ implies

$$\operatorname{supp}_\mathfrak{h} M_0 \subset \operatorname{supp}_\mathfrak{h} M^F_0 + \mathbb{Z}_{\geq 0}\Delta_{\mathfrak{h}\cap\mathfrak{g}_0[M]}(\mathfrak{g}_0[M]^I),$$

where $\Delta_{\mathfrak{h}\cap\mathfrak{g}_0[M]}(\mathfrak{g}_0[M]^I)$ stands for the set of roots of $\mathfrak{g}_0[M]^I$ (for the reader: check that $\Delta_{\mathfrak{h}\cap\mathfrak{g}_0[M]}(\mathfrak{g}_0[M]^I)$ is a subset of $\Delta_\mathfrak{h}$!). Hence by Proposition 8.5

$$\operatorname{supp}_\mathfrak{h} M = \operatorname{supp}_\mathfrak{h} M_0 + \Gamma_M \subset \operatorname{supp} M^F_0 + \bar{\Gamma}_M.$$

Therefore (8.4) holds. □

8.2.5 *Fundamental Results for Reductive* $\mathfrak{g}$

From this point on until the end of Sect. 8.2, we restrict ourselves to the case that $\mathfrak{g}$ is reductive and $\mathfrak{h}$ is a Cartan subalgebra. Under this assumption, every simple generalized weight module M is simply a weight module. Indeed, as $\mathfrak{h}$ acts semisimply on $\mathfrak{g}$, one can immediately check that for any nonzero generalized $\mathfrak{h}$-weight module M, the subspace $M' \subset M$ spanned by all $\mathfrak{h}$-eigenvectors in M (i.e., $m \in M$, such that $h \cdot m = \lambda(h)m$ for some $\lambda \in \mathfrak{h}^*$ and all $h \in \mathfrak{h}$) is itself a nonzero $\mathfrak{g}$-submodule.

The following theorem is proved in [DP1, Theorem 3] for a more general $\mathfrak{g}$, that is when $\mathfrak{g}$ is a root-reductive Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ satisfying certain conditions (see Sect. 8.3.2).

Theorem 8.13 *Let $\mathfrak{g}$ be a reductive Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let M be a simple $(\mathfrak{g}, \mathfrak{h})$-module. Then*

(1) $\mathfrak{g}_M^F\ \mathfrak{g}_M^I$, $\mathfrak{g}_M^+$, $\mathfrak{g}_M^-$ *are Lie subalgebras of $\mathfrak{g}$, and $\mathfrak{h}$ is a Cartan subalgebra for both $\mathfrak{g}_M^F$ and $\mathfrak{g}_M^I$.*
(2) $[(\mathfrak{g}_M^F)_{ss}, (\mathfrak{g}_M^I)_{ss}] = 0$ *and therefore $\mathfrak{g}_M^{FI} := \mathfrak{g}_M^F + \mathfrak{g}_M^I$ is a Lie subalgebra of $\mathfrak{g}$.*
(3) $\mathfrak{g}_M^+$ *and* $\mathfrak{g}_M^-$ *are $\mathfrak{g}_M^{FI}$-modules.*
(4) $\mathfrak{p}_M = \mathfrak{p}_M^+ := \mathfrak{g}_M^{FI} \oplus \mathfrak{g}_M^+$ *and* $\mathfrak{p}_M^- := \mathfrak{g}_M^{FI} \oplus \mathfrak{g}_M^-$ *are mutually opposite parabolic subalgebras of $\mathfrak{g}$.*

We come next to the following.

Theorem 8.14 (S.L. Fernando, V. Futorny) *Let M be a simple $(\mathfrak{g}, \mathfrak{h})$-module, where $\mathfrak{g}$ is reductive and $\mathfrak{h}$ is a Cartan subalgebra. Denote by M^+ the $\mathfrak{g}_M^+$-invariants of M. Then M^+ is a simple weight $\mathfrak{g}_M^{FI}$-module and the natural surjective map*

$$\pi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^+ \to M$$

is $\mathfrak{h}$-support preserving, i.e., $\operatorname{supp}_{\mathfrak{h}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^+) = \operatorname{supp}_{\mathfrak{h}} M$.

Proof Since $\mathfrak{g}_M^+$ is the nilradical of the parabolic subalgebra $\mathfrak{p}_M$, we have $[\mathfrak{p}_M, \mathfrak{g}_M^+] \subset \mathfrak{g}_M^+$, and hence M^+ is a $(\mathfrak{g}_M^F + \mathfrak{g}_M^I)$-module. Furthermore, the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^+$ is isomorphic as a $\mathfrak{g}_M^-$-module to $S^{\cdot}(\mathfrak{g}_M^-) \otimes_{\mathbb{C}} M^+$, and is graded by the total degree of elements of $S^{\cdot}(\mathfrak{g}_M)$. Moreover, the equality $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^+)^0 = M^+$ holds, where the superscript ^{0} stands for zero degree. This grading induces an ascending $\mathbb{Z}_{\leq 0}$-filtration on M, and using this filtration one verifies immediately that any proper $\mathfrak{p}_M$-submodule M' of M^+ yields a proper $\mathfrak{g}$-submodule $\pi^{-1}(M') \subset M$. Hence the irreducibility of M implies the irreducibility of M^+ as a $\mathfrak{p}_M$-module, i.e., M^+ is simple as a $\mathfrak{g}_M^F + \mathfrak{g}_M^I$-module. Finally, the statement about supports follows from Theorem 8.12, which implies that the supports of M and of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^+$ are equal. To the reader: show that $\operatorname{supp}_{\mathfrak{h}} U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^+$ equals the right-hand side of (8.4). □

Note that for $\mathfrak{g}$ and M as above, $\mathfrak{g}[M] = \mathfrak{g}_M^F + (\mathfrak{g}_M^I \cap \mathfrak{g}[M]) \supsetneq \mathfrak{g}_M^+$. In general, $\mathfrak{g}_M^I \cap \mathfrak{g}[M]$ is a root subalgebra of $\mathfrak{g}_M^I$ and does not have to be reductive. Indeed, the following is proved in [PS5].

Theorem 8.15 *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Any subalgebra $\mathfrak{k} \subset \mathfrak{g}$, such that $\mathfrak{k} \supset \mathfrak{h}$, is the Fernando–Kac subalgebra for a suitable simple $\mathfrak{h}$-weight $\mathfrak{g}$-module M.*

In the particular case when $\mathfrak{g} = \mathfrak{g}_M^I$, the subalgebra $\mathfrak{g}[M]$ can be any root subalgebra for which all root spaces $\mathfrak{g}^\alpha$ with $\mathfrak{g}^\alpha \nsubseteq \mathfrak{g}[M]$ generate $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{sl}(3)$, for instance the equality $\mathfrak{g} = \mathfrak{g}_M^I$ holds if and only if $\mathfrak{g}[M]$ equals either $\mathfrak{h}$ or $\mathfrak{h} \supsetneq \mathfrak{g}^\alpha$ for any root α.

In a general sense, Theorem 8.14 reduces the study of arbitrary simple weight modules to weight modules M with $\mathfrak{g} = \mathfrak{g}_M^I$. The first step in this reduction is to assign to M the $\mathfrak{g}_M^{FI}$-module M^+. The second (easy) step is provided by the following.

Lemma 8.16 *Fix a decomposition $\mathfrak{g}_M^{FI} = \mathfrak{a} \oplus [\mathfrak{g}_M^F, \mathfrak{g}_M^F] \oplus [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$. Then $M^+ \simeq (M^+)^F \boxtimes (M^+)^I$, where $(M^+)^F$ is a simple finite-dimensional $(\mathfrak{a} \oplus [\mathfrak{g}_M^F, \mathfrak{g}_M^F])$-module and $(M^+)^I$ is a simple $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-module that is an $(\mathfrak{h} \cap [\mathfrak{g}_M^I, \mathfrak{g}_M^I])$-weight module.*

Proof First note that M^+ is an irreducible $([\mathfrak{g}_M^F, \mathfrak{g}_M^F] \oplus [\mathfrak{g}_M^I, \mathfrak{g}_M^I])$-module. As $[\mathfrak{g}_M^F, \mathfrak{g}_M^F]$ acts integrably on M^+, we have that M^+ is isomorphic to a direct sum of (in general infinite-dimensional) isotypic components. Since each isotypic component is a module over $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ and since M^+ is a simple $\mathfrak{g}_M^{FI}$-module, it follows that M^+ is a single isotypic component. Hence, if M' is a simple $([\mathfrak{g}_M^F, \mathfrak{g}_M^F])$-submodule of this isotypic component, then

$$M^+ = M' \boxtimes \mathrm{Hom}_{[\mathfrak{g}_M^F, \mathfrak{g}_M^F]}(M', M^+).$$

Setting $(M^+)^F := M'$ when considered as a $(\mathfrak{a} \oplus [\mathfrak{g}_M^F, \mathfrak{g}_M^F])$ and letting $(M^+)^I := \mathrm{Hom}_{[\mathfrak{g}_M^F, \mathfrak{g}_M^F]}(M', M^+)$, we obtain the desired isomorphism. The $(\mathfrak{h} \cap [\mathfrak{g}_M^I, \mathfrak{g}_M^I])$-weight module structure on $(M^+)^I$ follows from the $\mathfrak{h}$-module structure on M^+. □

Let $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ denote the simple subalgebra of $\mathfrak{g}$ such that $\tilde{\mathfrak{g}}$ is a direct summand of the reductive part of a parabolic subalgebra of $\mathfrak{g}$. Lemma 8.16 makes it apparent that in order to classify simple weight $\mathfrak{g}$-modules it suffices to classify simple weight $\tilde{\mathfrak{g}}$-modules M' with $\tilde{\mathfrak{g}}[M'] = \tilde{\mathfrak{g}}_{M'}^I$, for all such subalgebras $\tilde{\mathfrak{g}} \subset \mathfrak{g}$. The latter problem however is within reach (at the current stage) only under the assumption of finite-dimensionality of the weight spaces. We discuss this below.

On a related note, we point out that Harish-Chandra modules provide important examples of simple weight modules M with $\mathfrak{g} = \mathfrak{g}_M^I$. Now let $\mathfrak{k} \subset \mathfrak{g}$ be a symmetric subalgebra of $\mathfrak{g}$ such that $\mathfrak{k} \supset \mathfrak{h}$, and let M be a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module without a highest weight (such modules are known to exist). Then $\mathfrak{k} \subset \mathfrak{g}[M]$.

Since it is well known that any intermediate subalgebra $\mathfrak{k} \subset \mathfrak{q} \subset \mathfrak{g}$ is parabolic, we infer that $\mathfrak{g}[M] = \mathfrak{k}$, or that $\mathfrak{g}[M]$ is parabolic. The second possibility is ruled out by the assumption that M is a non-highest weight $(\mathfrak{g}, \mathfrak{k})$-module. Hence $\mathfrak{g}[M] = \mathfrak{k}$. Next, since $\mathfrak{g}_M^+$ and $\mathfrak{g}_M^-$ are the nilpotent radicals of two mutually opposite parabolic subalgebras by Theorem 8.13*(4)*, we see that $\mathfrak{g}_M^+ = \mathfrak{g}_M^- = 0$ as otherwise $\mathfrak{g}_M^+, \mathfrak{h}, \mathfrak{g}_M^-$ would generate $\mathfrak{g}$ while the subalgebra generated by $\mathfrak{g}_M^+, \mathfrak{h}, \mathfrak{g}_M^-$ is a subalgebra of $\mathfrak{g}[M] = \mathfrak{k}$ by Theorem 8.1. In this way, we have $\mathfrak{g} = \mathfrak{g}_M^{FI}$, and now Theorem 8.13*(2)* implies that $\mathfrak{g} = \mathfrak{g}_M^I$ or $\mathfrak{g} = \mathfrak{g}_M^F$. Finally the possibility $\mathfrak{g} = \mathfrak{g}_M^F$ is excluded by the assumption that M is infinite dimensional, and we conclude that $\mathfrak{g} = \mathfrak{g}_M^I$.

In the next chapter we will discuss in more detail the Fernando–Kac subalgebra for $(\mathfrak{g}, \mathfrak{k})$-modules for general reductive subalgebras $\mathfrak{k}$ (generalized Harish-Chandra modules).

Proposition 8.17 (S.L. Fernando, [F]) *Let M be a $(\mathfrak{g}, \mathfrak{h})$-weight module with finite-dimensional weight spaces. Then $\mathfrak{g}[M] = \mathfrak{g}_M^F \supsetneq \mathfrak{g}_M^+$.*

Proof We need to show that $\mathfrak{g}_M^I \cap \mathfrak{g}[M] = \mathfrak{h}$. Fix a weight space M^κ and a root $\alpha \in (\Delta_\mathfrak{h})_M^I$. Assume to the contrary that $\mathfrak{g}^\alpha \subset \mathfrak{g}[M]$. Consider the $\mathfrak{sl}(2)^\alpha$-submodule $\bigoplus_{k\in\mathbb{Z}} M^{\kappa+k\alpha}$, where $\mathfrak{sl}(2)^\alpha$ is the $\mathfrak{sl}(2)$-root subalgebra of $\mathfrak{g}$ corresponding to α. Since $\alpha \in (\Delta_\mathfrak{h})_M^I$, we have $M^{\kappa+k\alpha} \neq 0$ for all $k \in \mathbb{Z}_{\geq 0}$. On the other hand, since $\mathfrak{g}^\alpha$ acts finitely on each weight space $M^{\kappa+k_0\alpha}$, for every $k_0 > 0$ there is $k_1 > k_0$ such that $M^{\kappa+k_1\alpha}$ contains a vector annihilated by $\mathfrak{g}^\alpha$. As k tends to infinity, these vectors generate infinitely many non-isomorphic $\mathfrak{sl}(2)$-constituents of $\oplus_k M^{\kappa+k\alpha}$, all of which have a 1-dimensional intersection with M^κ (standard theory of finite-dimensional $\mathfrak{sl}(2)$-modules). Hence $\dim M^\kappa = \infty$, a contradiction. □

This proposition makes it clear that in the case of finite-dimensional weight spaces, the $\mathfrak{h}$-shadow decomposition determines the Fernando–Kac subalgebra. As we pointed out in Sect. 8.2.3, this claim is false when weight spaces are infinite dimensional.

8.2.6 Cuspidal Modules and Mathieu's Coherent Families

Recall that $\mathfrak{g}$ is reductive and $\mathfrak{h}$ is a Cartan subalgebra, and let's assume in this section that the simple weight module M has finite-dimensional weight spaces. A simple weight $\mathfrak{g}$-module M is called *cuspidal*, if for any $\alpha \in \Delta_\mathfrak{h}$ and any $\kappa \in \operatorname{supp}_\mathfrak{h} M$, the map $\mathfrak{g}^\alpha \otimes M^\kappa \to M^{\kappa+\alpha}$ (induced by the $\mathfrak{g}$-module structure on M) is an isomorphism or, equivalently, if $\mathfrak{g} = \mathfrak{g}_M^I$. Lemma 8.16 implies that the $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-module $(M^+)^I$ is cuspidal as a $([\mathfrak{g}_M^I, \mathfrak{g}_M^I] \cap \mathfrak{h})$-weight module whenever $\mathfrak{g}_M^I \neq \mathfrak{h}$ (prove it!). In this way, the assignment

$$M \to M^+ = (M)^F \otimes_\mathbb{C} (M^+)^I$$

reduces the study of M to a tensor product of a finite-dimensional module over $\mathfrak{a} \oplus [\mathfrak{g}_M^F, \mathfrak{g}_M^F]$ with a cuspidal module over $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$. Let us discuss which root subalgebras can appear as $\mathfrak{g}_M^F$ and $\mathfrak{g}_M^I$ for a fixed $\mathfrak{h}$ and a variable M. First, the reader should prove that any parabolic subalgebra can appear as $\mathfrak{p}_M$ for some M. For this it suffices to fix a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and to show that there always exists a finite-dimensional simple $\mathfrak{p}$-module M^F such that $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^F$ is simple. (Prove this!) Then, clearly, $\mathfrak{p}_M = \mathfrak{p}$. It is a priori less clear which subalgebras can appear as $\mathfrak{g}_M^I$. Clearly, their Levi component should be a direct summand in the Levi component of a parabolic subalgebra, but is this sufficient? The following theorem of S.L. Fernando answers this question. Please read its proof in [F].

Theorem 8.18 *Let $\mathfrak{s}$ be a simple Lie algebra that admits a cuspidal simple weight module M with $\mathfrak{s} = \mathfrak{s}_M^I$. Then $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$ for some n.*

Theorem 8.18 finally reduces the classification of simple weight modules with finite weight multiplicities to the classification of simple cuspidal weight modules with finite weight multiplicities over the Lie algebras $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$. This latter classification was given by O. Mathieu in 1998 [Mat]. Here we will only try to give an idea of the depth and beauty of the problem, and will refer the reader to Mathieu's paper.

Assume for concreteness that $\mathfrak{g} = \mathfrak{sl}(n)$. Then a first straightforward question is whether there are *multiplicity free* simple cuspidal weight modules, i.e., simple cuspidal weight modules M with $\dim M^\lambda = 1$ whenever $M^\lambda \neq \{0\}$. (For $n = 2$ these modules correspond to case (4) in the example in Sect. 8.2.3.) The answer is "yes", and in fact it is not a trivial problem to classify all such simple cuspidal modules. After several partial results, this classification was obtained by D. Britten and F. Lemire in 1987 [BL]; here we simply state their result.

For each vector $a \in \mathbb{C}^n$ with $a_i \notin \mathbb{Z}$ for $i = 1, \dots, n$, let

$$N(a) := \operatorname{span}\{x^b := x_1^{b_1} \dots x_n^{b_n} \mid b_i - a_i \in \mathbb{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n} (b_i - a_i) = 0\}.$$

Then $\mathfrak{sl}(n)$ acts on $N(a)$ as derivations (the root space $\mathfrak{sl}(n)^{\varepsilon_i - \varepsilon_j}$ is spanned by the derivation $x_i \frac{\partial}{\partial x_j}$), and it is easy to see that $N(a)$ is a cuspidal simple multiplicity-free $\mathfrak{sl}(n)$-module. Indeed, it is clear that each root vector of $\mathfrak{sl}(n)$ transforms one monomial of the type $x_1^{b_1} \dots x_n^{b_n}$ into a multiple of another monomial, while the Cartan subalgebra of $\mathfrak{sl}(n)$ (lying in $\operatorname{span}\{x_i \frac{\partial}{\partial x_i} \mid i = 1, \dots, n\}$) transforms one monomial into a multiple of itself. Furthermore, the weights of all monomials are distinct, hence the $\mathfrak{sl}(n)$-module $N(a)$ is multiplicity free. Finally, as $b_i \notin \mathbb{Z}$, each pair of opposite root vectors $x_i \frac{\partial}{\partial x_j}$ and $x_j \frac{\partial}{\partial x_i}$ yields isomorphisms of the weight spaces of $N(a)^{(b_1,\dots,b_n)}$ and $N(a)^{(b_1,\dots,b_i-1,b_i+1,b_{i+1},\dots,b_{j-1},b_j-1,b_{j+1},\dots,b_n)}$ that are mutually inverse up to a nonzero constant. Hence, $N(a)$ is a simple cuspidal weight module.

The main result of [BL] concerning the case $\mathfrak{sl}(n)$ is the following.

Theorem 8.19 *Any simple cuspidal multiplicity free weight $\mathfrak{sl}(n)$-module is isomorphic to $N(a)$ for some $a \in \mathbb{C}^n$, $a_i \notin \mathbb{Z}$.*

Mathieu's classification of simple cuspidal weight modules with finite-dimensional spaces elevates the construction of the modules $N(a)$ to an elaborate theory. He considers semisimple *coherent families*, which by definition are (heavily reducible) semisimple weight modules $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ with finite-dimensional weight spaces such that

- $M^\lambda \neq 0$ for any $\lambda \in \mathfrak{h}^*$ and $\dim M^\lambda = const$;
- for any vector u in the centralizer $U(\mathfrak{h})^0$ of $\mathfrak{h}$ in $U(\mathfrak{g})$, the function

$$\lambda \mapsto \mathrm{tr}(u_{|M^\lambda})$$

is a polynomial in λ.

Mathieu's first main result is that any simple cuspidal weight module is an isotypic component (a "direct summand with multiplicity one") of a unique (up to isomorphism) semisimple coherent family. This reduces the classification of simple cuspidal weight modules to a classification of semisimple coherent families and to an understanding of their cuspidal simple isotypic components.

The key idea in classifying the semisimple coherent families is that they are determined by their highest weight simple direct summands. In fact, Mathieu proves that a semisimple coherent family always contains a simple highest weight module as a direct summand. This (in general non-unique) highest weight module is very degenerate as the dimensions of its weight spaces are bounded. Mathieu then classifies such bounded highest weight modules explicitly by determining their highest weights. Finally, given a bounded simple highest weight module L, its corresponding coherent family is constructed from L via (twisted) localization (one can localize modules $U(\mathfrak{g})$ with respect to multiplicative subsets of $U(\mathfrak{g})$). More precisely, Mathieu assigns to L a special multiplicative subset of $U(\mathfrak{g})$ generated by commuting root vectors.

This is a very brief outline of some of the ideas of the important paper [Mat]. The reader is encouraged to study it thoroughly. An important point is that Mathieu's techniques apply also to the case $\mathfrak{g} = \mathfrak{sp}(2n)$; therefore the classification of all simple cuspidal modules for all simple Lie algebras $\mathfrak{g}$ (by Theorem 8.18, $\mathfrak{g}$ is either $\mathfrak{sl}(n)$ or $\mathfrak{sp}(2n)$) can be carried out in reasonably general terms.

8.2.7 Weight Modules for Lie Superalgebras

We complete this section with a few words concerning the extension of the above results to Lie superalgebras.

A first advancement towards the classification of the simple weight modules with finite weight multiplicities over all simple finite-dimensional complex Lie superalgebras was made in 2000 in [DMP] where this classification was reduced to the classification of the so-called simple cuspidal modules. This result together with the classification result of Mathieu [Mat], for weight modules over Lie algebras, limited the consideration to the Lie superalgebras $\mathfrak{q}(n)$, $\mathfrak{osp}(m|n)$, $m = 1, 3, 4, 5, 6$, $D(2, 1; \alpha)$, and the Cartan-type series. On the other hand, with the aid of a twisted localization functor (introduced in [Mat] for Lie algebras and in [Gr2] for Lie superalgebras), the classification of the simple cuspidal modules over the Lie superalgebras in the latter list was reduced to the classification of all simple bounded highest weight modules. The simple bounded highest weight modules of $\mathfrak{q}(n)$ were classified in [GGr1], of $D(2, 1; \alpha)$ in [Hoy], of $\mathfrak{osp}(1|2n)$ in [FGGr], and of $\mathfrak{osp}(m|2n)$, for any m, n, in [GGr2]. Geometric construction of bounded weight $\mathfrak{sl}(n|1)$-modules, and more precisely of coherent families over $\mathfrak{sl}(n|1)$ were studied in [Gr1].

8.3 Weight Modules for Root-Reductive Lie Algebras

The theory of weight modules for root-reductive Lie algebras was first developed in the paper [DP1]. Below we present some highlights of the current stage of the theory.

8.3.1 Main Differences from the Finite-Dimensional Case

Let $\mathfrak{g}$ denote a root-reductive Lie algebra. Here is a theorem that does not hold for infinite-dimensional $\mathfrak{g}$.

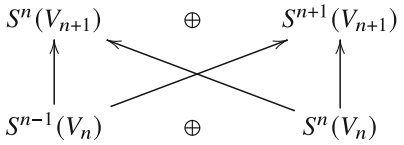
Theorem 8.20 *If $\mathfrak{g}$ is finite dimensional and M is a simple weight module, then either all weight spaces of M are finite dimensional or all weight spaces of M are infinite dimensional.*

Theorem 8.20 is a particular case of a much more general result [PS5, Lemma 6] for pairs $(\mathfrak{g}, \mathfrak{k})$, and our case is $\mathfrak{k} = \mathfrak{h}$.

Probably the simplest counterexample to Theorem 8.20 for an infinite-dimensional $\mathfrak{g}$ is the adjoint module of $\mathfrak{sl}(\infty)$. Here the zero weight space is infinite dimensional, while all other weight spaces have dimension one.

Next, if $\mathfrak{g}$ is finite dimensional, then any simple integrable module is necessarily finite dimensional, and hence a weight module. However, if $\mathfrak{g}$ is infinite dimensional, then an integrable simple module need not be a weight module. Consider the following example.

Example 8.21 Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ and fix an exhaustion of $\mathfrak{g}$ with $\mathfrak{g}_n \simeq \mathfrak{sl}(n)$. Set $M_n := S^{n-1}(V_n) \oplus S^n(V_n)$, and let $M = \varinjlim M_n$ where the embeddings $M_n \hookrightarrow M_{n+1}$ are given by the diagram



in which all arrows correspond to unique (up to multiplicative constants) embeddings of $\mathfrak{g}_n$-modules. Then if $\mathfrak{h}_n \subset \mathfrak{g}_n$, $\mathfrak{h}_{n+1} \subset \mathfrak{g}_{n+1}$ are Cartan subalgebras such that $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$, every $\mathfrak{h}_n$-weight space of M_n is mapped to two different $\mathfrak{h}_{n+1}$-weight spaces of M_{n+1}, and the reader can verify that this implies that M is not a weight module for any splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(\infty)$. On the other hand, M is clearly a simple integrable $\mathfrak{g}$-module as it is generated by any nonzero vector v and $\dim U(\mathfrak{g}_n)v < \infty$ for any n. Indeed, one verifies immediately that any $0 \neq v \in M_{n_0}$ for $n_0 \geq 2$ generates M_{n_0+k} over $\mathfrak{g}_{n_0+k}$ for any $k > 0$. ◯

Note that the module M from Example 8.21 has another interesting feature. Namely, it is not locally simple. We define a $\mathfrak{g}$-module to be locally simple with respect to a fixed exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ if $M = \varinjlim M_n$ for simple $\mathfrak{g}_n$-modules M_n. Obviously, a locally simple module is necessarily simple. Moreover, if M is a locally simple $\mathfrak{g}$-module, then any $v \in M$ generates a simple $\mathfrak{g}_n$-submodule for $n >> 0$. In Example 8.21, any nonzero vector $v \in M$ generates a $\mathfrak{g}_n$-submodule of length two for $n >> 0$. Therefore this module M cannot be locally simple with respect to any fixed exhaustion. It is a bit harder, but still possible to prove that M is not locally simple for any exhaustion of $\mathfrak{g} = \mathfrak{sl}(\infty)$ by simple Lie algebras $\mathfrak{g}_n$.

Finally, let's discuss when a simple integrable weight module is a highest weight module. If $\mathfrak{g}$ is finite dimensional, and the $\mathfrak{g}$-module M is simple and integrable, then it is a highest weight module with respect to every Borel subalgebra. This claim is clearly false for infinite-dimensional $\mathfrak{g}$, as not even the natural module of $\mathfrak{sl}(\infty)$ is a highest weight module with respect to every Borel subalgebra. (For the reader: determine for which Dynkin Borel subalgebra(s) in Example 5.6 the natural module has a highest weight.)

Every tensor module of $\mathfrak{sl}(\infty)$ is a highest weight module with respect to the perfect Borel subalgebra (5.4). Can we have a locally simple, integrable weight module that has no highest weight with respect to any Borel subalgebra? This question was answered affirmatively in [DP1] by giving the following example.

Example 8.22 Let V_n denote the natural module of $\mathfrak{sl}(n)$. Consider the module embedding $S^{n-1}(V_n) \to S^n(V_{n+1})$ given by the left vertical arrow in Example 8.21. One can check that this embedding sends the extremal weights $(0, \cdots, 0, n-1, 0, \cdots, 0)$ of $S^{n-1}(V_n)$ to the weights $(0, \cdots, 0, n-1, 0, \cdots, 0, 1)$ of $S^n(V_{n+1})$, which are not extremal for $S^n(V_{n+1})$. We let M be the module obtained by taking the direct limit. Then M is a locally simple, integrable weight $\mathfrak{sl}(\infty)$-module, with one-

dimensional weight spaces, but it has no highest weight with respect to any Borel subalgebra containing $\mathfrak{h}$. Geometrically, this corresponds to embedding simplices in such a way that the vertices of the smaller simplex lie on edges of the larger simplex but not on the vertices. Hence, the limit of these polytopes is a geometric object with no vertices. ◯

8.3.2 The Shadow Decomposition

Henceforth we will restrict our attention to weight modules over a root-reductive Lie algebra $\mathfrak{g}$. By Theorem 3.1, the commutator subalgebra $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is isomorphic to a direct sum of copies of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ and finite-dimensional simple Lie algebras (each with at most countable multiplicity), and $\mathfrak{g}$ is an extension of an abelian Lie algebra $\mathfrak{a}$ by $\mathfrak{g}'$.

We let $\mathfrak{t}'$ be a maximal toral subalgebra of $\mathfrak{g}'$ that is splitting (i.e., $\mathfrak{g}'$ decomposes into weight spaces over $\mathfrak{t}'$). It follows that $\mathfrak{t}'$ is a (splitting) Cartan subalgebra of $\mathfrak{g}'$ since $\mathfrak{t}' = C_{\mathfrak{g}'}(\mathfrak{t}')$. We extend $\mathfrak{t}'$ to a maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. We fix the Cartan subalgebra $\mathfrak{h} := C_{\mathfrak{g}}(\mathfrak{t})$ of $\mathfrak{g}$.

We first claim that $\mathfrak{h} = \mathfrak{t}$, that is $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$. Indeed, by the proof of [DPSn, Theorem 4.12], we have $C_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t} + C_{\mathfrak{g}'}(\mathfrak{t})$. As

$$C_{\mathfrak{g}'}(\mathfrak{t}) \subset C_{\mathfrak{g}'}(\mathfrak{t}') = \mathfrak{t}' \subset \mathfrak{t},$$

we conclude $C_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$. Now we will show that $\mathfrak{h}$ is splitting. Indeed, $\mathfrak{g}$ decomposes into generalized weight spaces over $\mathfrak{h}$, since $\mathfrak{h}$ is maximal toral. The nonzero weight spaces of $\mathfrak{g}$ over $\mathfrak{h}$ coincide with the nonzero weight spaces of $\mathfrak{g}'$ over $\mathfrak{h}'$ and are hence one dimensional by the classification of simple root-reductive Lie algebras. Moreover, by [DPSn, Theorem 4.12], $\mathfrak{h} = \mathfrak{g}_{\mathfrak{h}}^0$. Hence, all generalized weight spaces are in fact weight spaces, which means that $\mathfrak{h}$ is a splitting Cartan subalgebra.

Under the above assumptions on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the notion of shadow decomposition was extended to root-reductive Lie algebras in [DP1]. The definitions of $(\Delta_{\mathfrak{h}})_M^F$, $(\Delta_{\mathfrak{h}})_M^I$, $(\Delta_{\mathfrak{h}})_M^+$, $(\Delta_{\mathfrak{h}})_M^-$ are exactly the same as for finite-dimensional reductive Lie algebras (see Sect. 8.2.2). And just as before, the $\mathfrak{h}$-shadow decomposition of M is defined to be

$$\mathfrak{g} = (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+ \oplus \mathfrak{g}_M^-. \tag{8.5}$$

It is proved in [DP1] that the $\mathfrak{h}$-shadow decomposition of M is well defined, and moreover that Theorem 8.13 holds in this setting. An important corollary of the following theorem from [DP1] is that for every parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, there exists an irreducible $\mathfrak{g}$-module M, such that $\mathfrak{p} = \mathfrak{p}_M^+$ where $\mathfrak{p}_M^+$ is the parabolic subalgebra as defined in Theorem 8.13.

Theorem 8.23 *Let $\mathfrak{g}$ be a root-reductive Lie algebra with Cartan subalgebra $\mathfrak{h}$ as above. For any given splitting*

$$\Delta = \Delta^F \sqcup \Delta^I \sqcup \Delta^+ \sqcup \Delta^-$$

with $\Delta^F = -\Delta^F$, $\Delta^I = -\Delta^I$, $\Delta^- = -\Delta^+$, such that its corresponding decomposition

$$\mathfrak{g} = (\mathfrak{g}^F + \mathfrak{g}^I) \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^-, \tag{8.6}$$

satisfies the properties (1)–(3) of Theorem 8.13, there exists an irreducible weight module M for which (8.6) is the $\mathfrak{h}$-shadow decomposition of $\mathfrak{g}$ with respect to M.

Next we point out that Proposition 8.17 holds for any root-reductive Lie algebra $\mathfrak{g}$ (with $\mathfrak{h}$ as above) by the same proof as before, and this follows from the fact that when M has finite-dimensional weight spaces, $\mathfrak{g}[M] \cap \mathfrak{g}_M^I = \mathfrak{h}$. Indeed, for any irreducible weight module M (with possibly infinite-dimensional weight spaces) over a root-reductive Lie algebra $\mathfrak{g}$, the Fernando–Kac subalgebra is always given by the formula

$$\mathfrak{g}[M] = (\mathfrak{g}_M^F \oplus (\mathfrak{g}[M] \cap \mathfrak{g}_M^I)) \supsetneq \mathfrak{g}_M^+.$$

If $\mathfrak{g}$ is simple and finite dimensional, then we cannot have $\mathfrak{g}[M] = \mathfrak{g}_M^I$ (prove this!). However, if $\mathfrak{g} = \mathfrak{sl}(\infty)$, then it is possible that $\mathfrak{g} = \mathfrak{g}[M] = \mathfrak{g}_M^I$ (recall that $\mathfrak{g} = \mathfrak{g}[M]$ if and only if M is integrable). In fact, if $\mathfrak{g}$ is simple and infinite dimensional, then for any (irreducible) integrable weight module M, we have either $\mathfrak{g} = \mathfrak{g}_M^I$ or $\mathfrak{g} = \mathfrak{g}_M^F$, and both cases are possible [DP1, Proposition 6].

Recall that if $\mathfrak{g} = \mathfrak{g}_M^I$, then M is called cuspidal. Now if $\mathfrak{g} = \mathfrak{g}_M^F$, then we will call M *finite-integrable*. Question: can a finite-integrable module have infinite-dimensional weight spaces? Answer: yes, consider for example the adjoint module of $\mathfrak{sl}(\infty)$. Here the zero weight space is infinite dimensional. However, if an irreducible integrable $(\mathfrak{g}, \mathfrak{h})$-module has finite-dimensional weight spaces, then it is necessarily finite-integrable (prove!).

The following theorem is proved in [DP1].

Theorem 8.24 *Let M be an irreducible integrable $(\mathfrak{g}, \mathfrak{h})$-module with finite-dimensional weight spaces. Then*

(1) M is locally simple (with respect to any exhaustion of $\mathfrak{g}$ as $\varinjlim \mathfrak{g}_n$ where $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_{n+1}$ are root embeddings for Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{g}_n$, $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$, $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$ such that $\varinjlim \mathfrak{h}_n = \mathfrak{h}$).
(2) $\operatorname{supp} M$ determines M up to isomorphism.

Note that Theorem 8.24 appears in [DP1] as Theorem 6, where it is stated under the weaker condition that M is finite. However, the condition that the weight spaces of M are finite dimensional is needed for the proof given in [DP1].

8.3.3 Simple Bounded Weight Modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$

Recall that a weight module is called bounded if the dimensions of all its weight spaces are bounded by a constant depending on the module. A classification of simple bounded weight modules for $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ recently appeared in [GrP]. Here we highlight some key results from this work.

Theorem 8.25 *Every simple bounded weight* $\mathfrak{g}$*-module is locally simple.*

The proof of this theorem given in [GrP] is a bit technical and uses essentially the work of Mathieu [Mat], so we will not reproduce it here.

We first discuss the classification of integrable simple bounded weight modules. Let's start with some definitions.

A subset $\mathbb{A}$ of $\mathbb{Z}_{>0}$ is *semi-infinite* if both $\mathbb{A}$ and $\mathbb{Z}_{>0} \setminus \mathbb{A}$ are infinite. For two semi-infinite sets $\mathbb{A}_1$ and $\mathbb{A}_2$ of $\mathbb{Z}_{>0}$, we write $\mathbb{A}_1 \approx \mathbb{A}_2$ if there exist disjoint finite subsets $F_{\mathbb{A}_1}$ and $F_{\mathbb{A}_2}$ of $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively, so that $F_{\mathbb{A}_1}$ and $F_{\mathbb{A}_2}$ are of equal cardinality and $\mathbb{A}_1 \setminus F_{\mathbb{A}_1} = \mathbb{A}_2 \setminus F_{\mathbb{A}_2}$. Obviously, $\approx$ defines an equivalence relation on the set of semi-infinite subsets of $\mathbb{Z}_{>0}$.

For a semi-infinite subset $\mathbb{A}$ of $\mathbb{Z}_{>0}$, let k_n be the number of elements in $\mathbb{A} \cap [1, n]$. Consider the k_n-th exterior power $\Lambda^{k_n}(V_{n+1})$ of the natural representation V_{n+1} of $\mathfrak{sl}(n+1)$. Since $k_n \leq n$, we have $\Lambda^{k_n}(V_{n+1}) \neq 0$. Moreover, there is a unique up to a multiplicative constant monomorphism of $\mathfrak{sl}(n+1)$-modules $\Lambda^{k_n}(V_{n+1}) \hookrightarrow \Lambda^{k_{n+1}}(V_{n+2})$. The resulting direct limit $\varinjlim \Lambda^{k_n}(V_{n+1})$ will be denoted by $\Lambda^{\frac{\infty}{2}}_{\mathbb{A}} V$, and we call it a *semi-infinite fundamental representation* of $\mathfrak{g} = \mathfrak{sl}(\infty)$. It is straightforward to check that $\Lambda^{\frac{\infty}{2}}_{\mathbb{A}_1} V \simeq \Lambda^{\frac{\infty}{2}}_{\mathbb{A}_2} V$ if and only if $\mathbb{A}_1 \approx \mathbb{A}_2$ (the reader should do it!).

Given a splitting Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(\infty)$ corresponding to a strict linear order $\prec$ on $\mathbb{Z}_{>0}$, we say that $\mathfrak{b}$ is a $\mathbb{A}$-*compatible* for a subset $\mathbb{A} \subset \mathbb{Z}_{>0}$, if $a \prec b$ for all $a \in \mathbb{A}$, $b \in \mathbb{Z}_{>0} \setminus \mathbb{A}$.

Next, fix an infinite nondecreasing sequence $\mathbb{A} = \{a_1, a_2, \ldots \mid a_i \leq a_{i+1}\}$ of positive integers with $\lim a_n = \infty$. There is a unique, up to a multiplicative constant, monomorphism of $\mathfrak{g}_n$-modules $S^{a_n}(V_{n+1}) \hookrightarrow S^{a_{n+1}}(V_{n+2})$. The resulting direct limit $\varinjlim S^{a_n}(V_{n+1})$ will be denoted by $S^{\infty}_{\mathbb{A}} V$. This series of modules generalizes Example 8.22 where $M = S^{\infty}_{\mathbb{A}} V$ with $\mathbb{A} = (1, 2, 3, \ldots)$.

For two nondecreasing sequences $\mathbb{A}_1$ and $\mathbb{A}_2$ as above, we write $\mathbb{A}_1 \sim \mathbb{A}_2$ if $a_n = b_n$ for all n greater than some n_0. The reader can check that there is an isomorphism of $\mathfrak{sl}(\infty)$-modules $S^{\infty}_{\mathbb{A}_1} V \simeq S^{\infty}_{\mathbb{A}_2} V$ if and only if $\mathbb{A}_1 \sim \mathbb{A}_2$.

Similarly, for a sequence $\mathbb{A}$ as above, we introduce the $\mathfrak{sl}(\infty)$-modules $S^{\infty}_{\mathbb{A}} V_*$.

We are now ready for the following.

Theorem 8.26 *Let M be a simple integrable bounded weight module of $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then M is isomorphic to one of the following:*

(1) $\Lambda^{\frac{\infty}{2}}_{\mathbb{A}} V$ for a semi-infinite subset $\mathbb{A} \subset \mathbb{Z}_{>0}$,
(2) $S^{\infty}_{\mathbb{A}} V$ for an infinite sequence $\mathbb{A}$ as above,

(3) $S^{\infty}_{\mathbb{A}} V_*$ *for an infinite sequence* $\mathbb{A}$ *as above,*
(4) $V_{\eta,\emptyset}$ *or* $V_{\emptyset,\eta}$ *for a partition* η*,*

where $V_{\lambda,\mu}$ *denotes a simple tensor module (see Sect. 7.1), and* $\emptyset$ *stands for the empty partition. All isomorphisms between modules from the above list are:* $\Lambda^{\frac{\infty}{2}}_{\mathbb{A}_1} V \simeq \Lambda^{\frac{\infty}{2}}_{\mathbb{A}_2} V$ *for* $\mathbb{A}_1 \approx \mathbb{A}_2$, $S^{\infty}_{\mathbb{A}_1} V \simeq S^{\infty}_{\mathbb{A}_2} V$ *and* $S^{\infty}_{\mathbb{A}_1} V_* \simeq S^{\infty}_{\mathbb{A}_2} V_*$ *for* $\mathbb{A}_1 \sim \mathbb{A}_2$.

Let's discuss some features of the modules classified in Theorem 8.26. Our first observation is that the modules $\Lambda^{\frac{\infty}{2}}_{\mathbb{A}} V$, $S^{\infty}_{\mathbb{A}} V$ and $S^{\infty}_{\mathbb{A}} V_*$ are multiplicity free. This can be checked easily by using the fact that they are direct limits of multiplicity free $\mathfrak{h}_n$-modules (for the reader: check that all symmetric and exterior powers of the natural representation of $\mathfrak{sl}(n)$ are multiplicity free weight modules!). The modules in *(4)* are multiplicity free only in the cases where the partition η consists of a single part, all parts of η equal 1, or $\eta = \emptyset$.

Another natural question to ask about an integrable module is whether it is a highest weight module with respect to some splitting Borel subalgebra. For the modules *(1)*–*(4)*, the answer turns out to be that the modules in *(1)* and *(4)* are highest weight with respect to a particular choice of Borel subalgebra, while the modules in *(2)* and *(3)* are not highest weight modules with respect to any Borel subalgebra. For *(2)* and *(3)* this is more or less for the same reason as given in Example 8.22, and the reader could carefully check this as an exercise. In the case of *(4)*, all modules are highest weight modules with respect to a perfect Borel subalgebra, as defined in Example 5.4. A Dynkin Borel subalgebra will also work but it will depend on the module, that is for $V_{\eta,\emptyset}$ we can take the strict linear order $1 \prec 2 \prec 3 \prec 4 \prec \cdots$, while for $V_{\emptyset,\eta}$ we can take $\cdots \prec 4 \prec 3 \prec 2 \prec 1$. All modules from *(1)* are highest weight modules with respect to some Dynkin Borel subalgebra, whose choice depends on the module. However this linear order is always isomorphic to $\cdots \prec 6 \prec 4 \prec 2 \prec 1 \prec 3 \prec 5 \prec \cdots$ (prove!).

Next we turn to the case of $\mathfrak{o}(\infty)$. In the following discussion, either $\mathfrak{g}_n = \mathfrak{o}(2n+1)$ for all n or $\mathfrak{g}_n = \mathfrak{o}(2n)$ for all n.

Proposition 8.27 *Let* M *be a nontrivial bounded simple weight* $\mathfrak{o}(\infty)$*-module. Then* M *is necessarily integrable, is multiplicity free, and is isomorphic to* V *or to a direct limit of (simple) spinor modules of* $\mathfrak{g}_n$*.*

In order to make this result more explicit, we would like to construct direct limits of spinor modules. We fix the Borel subalgebra $\mathfrak{b}_n$ with positive roots

$$\{\varepsilon_i - \varepsilon_j, \varepsilon_k + \varepsilon_\ell, \varepsilon_m \mid i < j, k < \ell\}$$

if $\mathfrak{g}_n = \mathfrak{o}(2n+1)$, and

$$\{\varepsilon_i - \varepsilon_j, \varepsilon_k + \varepsilon_\ell \mid i < j, k < \ell\}$$

if $\mathfrak{g}_n = \mathfrak{o}(2n)$. Recall that the Lie algebra $\mathfrak{g}_n = \mathfrak{o}(2n+1)$ has (up to isomorphism) one spinor representation $\mathcal{S}_n$, and it has $\mathfrak{b}_n$-highest weight $\left(\frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2}\right)$. It is a

standard fact that, up to a multiplicative constant, there are exactly two weight embeddings of $\mathcal{S}_n$ into $\mathcal{S}_{n+1}$, which will be denoted by ι_n^+ and ι_n^-. Under the embedding $\iota_n^\pm$, the vectors of weight $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ are mapped to vectors of weight $\left(\frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2}\right)$. Next, recall that the Lie algebra $\mathfrak{g}_n = \mathfrak{o}(2n)$ has (up to isomorphism) two spinor representations, $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, with respective $\mathfrak{b}_n$-highest weights $\left(\frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}\right)$. In this case, it is easy to verify that there are unique, up to multiplicative constants, weight embeddings $\mathcal{S}_n^+ \hookrightarrow \mathcal{S}_{n+1}^+$, $\mathcal{S}_n^+ \hookrightarrow \mathcal{S}_{n+1}^-$, $\mathcal{S}_n^- \hookrightarrow \mathcal{S}_{n+1}^+$ and $\mathcal{S}_n^- \hookrightarrow \mathcal{S}_{n+1}^-$.

Recall from Proposition 7.12 that $\mathfrak{o}(\infty)$ has two conjugacy classes of splitting Cartan subalgebras. We will denote our fixed Cartan subalgebra $\mathfrak{h}$ by $\mathfrak{h}_B$ if the root system Δ corresponding to $\mathfrak{h}$ is of type (5.6), and by $\mathfrak{h}_D$ if the root system Δ corresponding to $\mathfrak{h}$ is of type (5.7).

Definition 8.28 Let $\mathbb{A}$ be a subset of $\mathbb{Z}_{>0}$.

- By $\mathcal{S}_{\mathbb{A}}^B$ we denote the $\mathfrak{h}_B$-weight $\mathfrak{g}$-module obtained as the direct limit of the weight embeddings $\varphi_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ defined by $\varphi_n = \iota_n^+$ if $n \in \mathbb{A}$ and $\varphi_n = \iota_n^-$ if $n \notin \mathbb{A}$.
- By $\mathcal{S}_{\mathbb{A}}^D$ we denote the $\mathfrak{h}_D$-weight $\mathfrak{g}$-module obtained as the direct limit of weight embeddings $\varphi_n : M_n \to M_{n+1}$ such that $M_n = \mathcal{S}_n^+$ if $n \in \mathbb{A}$ and $M_n = \mathcal{S}_n^-$ if $n \notin \mathbb{A}$.

To a subset $\mathbb{A} \subset \mathbb{Z}_{>0}$ we assign weights $\omega_{\mathbb{A}}^B \in \mathfrak{h}_B^*$ and $\omega_{\mathbb{A}}^D \in \mathfrak{h}_D^*$ by putting $(\omega_{\mathbb{A}}^B)_k = (\omega_{\mathbb{A}}^D)_k := \frac{1}{2}$ if $k \in \mathbb{A}$, $(\omega_{\mathbb{A}}^B)_k = (\omega_{\mathbb{A}}^D)_k := -\frac{1}{2}$ otherwise. For two weights $\lambda, \mu \in \mathfrak{h}_B^*$ we write $\lambda \sim_B \mu$ if $\lambda_i = \mu_i$ for all but finitely many indices $i > 0$. For two weights $\lambda, \mu \in \mathfrak{h}_D^*$ we write $\lambda \sim_D \mu$ if $\lambda_i \neq \mu_i$ for an even (in particular, finite) number of indices $i > 0$.

The following theorem, together with Proposition 8.27, completes the classification of simple bounded $\mathfrak{o}(\infty)$-modules.

Theorem 8.29 *Let M be a simple weight module that is a direct limit of spinor modules of $\mathfrak{o}(2n+1)$ or $\mathfrak{o}(2n)$ for $n \to \infty$. Then $M \simeq \mathcal{S}_{\mathbb{A}}^B$ or $M \simeq \mathcal{S}_{\mathbb{A}}^D$ for some $\mathbb{A} \subset \mathbb{Z}_{>0}$. Furthermore, $\mathcal{S}_{\mathbb{A}_1}^B \simeq \mathcal{S}_{\mathbb{A}_2}^B$ if and only if $\mathbb{A}_1 \sim \mathbb{A}_2$.*

The modules $S_{\mathbb{A}}^B$ and $S_{\mathbb{A}}^D$ are multiplicity free, and the reader can check that each of them is a highest weight module with respect to some splitting Borel subalgebra depending on the choice of $\mathbb{A}$. In fact, in the case of $\mathfrak{o}(\infty)$ the spinor modules $S_{\mathbb{A}}^B$ and $S_{\mathbb{A}}^D$ are natural analogs of the semi-infinite fundamental representations $\Lambda_{\mathbb{A}}^{\frac{\infty}{2}} V$ of $\mathfrak{sl}(\infty)$.

Finally, we have the following surprisingly simple statement for $\mathfrak{sp}(\infty)$.

Proposition 8.30 *Let M be a nontrivial bounded simple integrable weight $\mathfrak{sp}(\infty)$-module. Then M is isomorphic to the natural $\mathfrak{sp}(\infty)$-module V.*

It remains for us to give the classification of the non-integrable modules, but thanks to Proposition 8.27, we only need to consider $\mathfrak{g} = \mathfrak{sl}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$.

We start with some notation. For a finite or infinite sequence $\mu = (\mu_1, \mu_2, ...)$ of complex numbers, by μ_{int}, respectively, by μ_+ or μ_-, we denote the subset of $\mathbb{Z}_{>0}$ consisting of all i such that $\mu_i \in \mathbb{Z}$, respectively, $\mu_i \in \mathbb{Z}_{\geq 0}$ or $\mu_i \in \mathbb{Z}_{<0}$. If μ is a finite sequence, we set $|\mu| := \sum_{i>0} \mu_i$.

Now we recall the standard homomorphisms of $U(\mathfrak{g}) \to \mathcal{D}(\infty)$, where $\mathcal{D}(\infty)$ is the algebra of polynomial differential operators of $\mathbb{C}[x_1, x_2, ...]$ with infinitely many formal variables $x_1, x_2 \ldots$. For $\mathfrak{g} = \mathfrak{sl}(\infty)$, this standard homomorphism is given by $e_{ij} \mapsto x_i \partial_j$, whereas for $\mathfrak{g} = \mathfrak{sp}(\infty)$, it is given by $e_{\varepsilon_i+\varepsilon_j} \mapsto x_i x_j$ if $i \neq j$, $e_{2\varepsilon_i} \mapsto \frac{1}{2} x_i^2$, $e_{-\varepsilon_i-\varepsilon_j} \mapsto -\partial_i \partial_j$ if $i \neq j$, $e_{-2\varepsilon_i} \mapsto -\frac{1}{2}\partial_i^2$, where $e_\alpha \in \mathfrak{g}_n^\alpha$ are appropriate nonzero vectors. This enables us to realize $\mathcal{D}(\infty)$-modules as $U(\mathfrak{g})$-modules.

Next, we define a $\mathcal{D}(\infty)$-module as follows. We set $x^\mu := x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3} \cdots$ for any infinite sequence $\mu = (\mu_1, \mu_2, \mu_3...)$ of complex numbers. For a fixed sequence μ, the space of shifted Laurent polynomials

$$\mathcal{F}(\mu) := \{x^\mu p \mid p \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ...]\}$$

is a $\mathcal{D}(\infty)$-module in a natural way, and by the above embedding, $\mathcal{F}(\mu)$ is naturally a module over $\mathfrak{g} = \mathfrak{sl}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$. We single out an $\mathfrak{sl}(\infty)$-submodule and an $\mathfrak{sp}(\infty)$-submodule:

$$\mathcal{F}_{\mathfrak{sl}}(\mu) := \{x^\mu p \mid p \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ...], \deg p = 0\}$$

and

$$\mathcal{F}_{\mathfrak{sp}}(\mu) := \{x^\mu p \mid p \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ...], \deg p \in 2\mathbb{Z}\}.$$

We first concentrate on the case of $\mathfrak{sl}(\infty)$ and consider the module $\mathcal{F}_{\mathfrak{sl}}(\mu)$; it is a multiplicity free $\mathfrak{sl}(\infty)$-module. We set

$$V_{\mathfrak{sl}}(\mu) := \mathrm{Span}\{x^\lambda \mid \lambda - \mu \in Q_{\mathfrak{gl}(\infty)}, \mu_+ \subset \lambda_+\},$$

where $Q_{\mathfrak{gl}(\infty)}$ is the root lattice of $\mathfrak{gl}(\infty)$. Then we have $V_{\mathfrak{sl}}(\mu') \subset V_{\mathfrak{sl}}(\mu)$ if and only if $\mu - \mu' \in Q_{\mathfrak{gl}(\infty)}$ and $\mu_+ \subset \mu'_+$. Furthermore, we let $V_{\mathfrak{sl}}(\mu)^+ := 0$ whenever $\mu_+ = \mu_{int}$. If $\mu_+ \subsetneq \mu_{int}$, we define

$$V_{\mathfrak{sl}}(\mu)^+ := \sum_{V_{\mathfrak{sl}}(\mu') \subsetneq V_{\mathfrak{sl}}(\mu)} V_{\mathfrak{sl}}(\mu').$$

Finally, we define the $\mathfrak{sl}(\infty)$-module $X_{\mathfrak{sl}}(\mu)$ by setting

$$X_{\mathfrak{sl}}(\mu) := V_{\mathfrak{sl}}(\mu)/V_{\mathfrak{sl}}(\mu)^+.$$

In order to state the classification theorem for simple bounded non-integrable weight $\mathfrak{sl}(\infty)$-modules, we need one more definition. If μ and μ' are infinite sequences of complex numbers, then we write $\mu \sim_{\mathfrak{sl}} \mu'$ if $\mu - \mu' \in Q_{\mathfrak{gl}(\infty)}$ and $\mu_+ = \mu'_+$, where $Q_{\mathfrak{gl}(\infty)}$ denotes the root lattice of $\mathfrak{gl}(\infty)$ and we consider its elements as infinite sequences via the identification

$$\sum_{i=1}^{n} c_i \varepsilon_i \to (c_1, \ldots, c_n, 0, 0, 0, \ldots).$$

We reach the following.

Theorem 8.31

(1) Every simple bounded non-integrable $\mathfrak{sl}(\infty)$-module M is isomorphic to $X_{\mathfrak{sl}}(\mu)$ for some infinite sequence μ. In particular, $M \simeq \varinjlim X_{\mathfrak{sl}}(\mu^n)$, where $\mu^n := (\mu_1, ..., \mu_n)$.

(2) $X_{\mathfrak{sl}}(\mu) \simeq X_{\mathfrak{sl}}(\mu')$ if and only if $\mu \sim_{\mathfrak{sl}} \mu'$ or if $\mu = (0, 0, \ldots)$ and $\mu' = (-1, -1, -1, \ldots)$, or vice versa.

Theorem 8.31 has the following striking implication.

Corollary 8.32 *Every simple bounded non-integrable $\mathfrak{g}$-module is multiplicity free.*

This corollary is false for $\mathfrak{sl}(n)$. The classification of multiplicity free simple $\mathfrak{sl}(n)$-modules goes by the work of Britten and Lemire, and Mathieu's main achievement in his paper [Mat] is the classification of simple bounded weight $\mathfrak{sl}(n)$-modules beyond the multiplicity free case. The work of Fernando [F] is foundational for the work of Mathieu. Soon after the preprint of Mathieu's paper appeared in 1998, I. Dimitrov suggested that all simple bounded non-integrable weight $\mathfrak{sl}(\infty)$-modules should be multiplicity free, but the proof only appeared 20 years later in [GrP].

The modules $X_{\mathfrak{sl}}(\mu)$ have interesting structure, and are not highest weight modules in general. A detailed study of their structure has been initiated in [GrP] and further carried out in [Cal]. In particular, these modules can be cuspidal as well as integrable, and the latter occurs if and only if either all of the μ_i are non-negative integers, or all μ_i are strictly negative integers.

Assume that are all μ_i are non-negative. Then there are two subcases. If the sum of the μ_i is finite, then $X_{\mathfrak{sl}}(\mu)$ is isomorphic to $S^m(V)$ for $m = \sum \mu_i$. If the sum is infinite, then $X_{\mathfrak{sl}}(\mu)$ is isomorphic to $S^{\infty}_{\mathbb{A}} V$ for $\mathbb{A} = \{\mu_1, \mu_1 + \mu_2, \ldots\}$. If all of the μ_i are strictly negative, the respective subcases are: $\mu_i = -1$ for all but finitely many indices i, or $\mu_i < -1$ for infinitely many indices i. In the first subcase, there is an isomorphism $X_{\mathfrak{sl}}(\mu) \simeq S^m(V_*)$ for $m = \sum_{i>0}(-1-\mu_i)$. In the second subcase, we have $X_{\mathfrak{sl}}(\mu) \simeq S^{\infty}_{\mathbb{A}} V_*$ for $\mathbb{A} = \{a_1 = -1-\mu_1, ..., a_i = -i-\mu_1-\cdots-\mu_i, ...\}$.

Next we want to describe the sequences μ for which $X_{\mathfrak{sl}}(\mu)$ is a highest weight module. This is of course equivalent to describing all simple highest weight $\mathfrak{sl}(\infty)$-modules that are bounded.

It turns out that a simple bounded highest weight module is always a highest weight module with respect to a Dynkin Borel subalgebra. The module $X_{\mathfrak{sl}}(\mu)$ is a highest weight module whenever μ has at most one entry a different from 0 and -1. More precisely, under this assumption, $X_{\mathfrak{sl}}(\mu)$ is a highest weight module with respect to a Borel subalgebra corresponding to a strict linear order on the indices i such that all indices with $\mu_i = -1$ come first, and all indices with $\mu_i = 0$ come last. Moreover, $X_{\mathfrak{sl}}(\mu')$ is a highest weight module if and only if $\mu' \sim_{\mathfrak{sl}} \mu$.

The final point in our discussion of simple bounded weight $\mathfrak{sl}(\infty)$-modules is that a module $X_{\mathfrak{sl}}(\mu)$ may be a direct limit of simple highest weight $\mathfrak{sl}(n)$-modules, and yet not be a highest weight $\mathfrak{sl}(\infty)$-module. Such a module is by definition a *pseudo highest weight module*. First examples of pseudo highest modules are the modules $S^{\infty}_{\mathbb{A}} V$, which are direct limits of finite-dimensional simple $\mathfrak{sl}(n)$-modules, and thus of highest weight modules. A more interesting example of a pseudo highest weight module with respect to the Borel subalgebra corresponding to the strict linear order $\cdots \prec 6 \prec 4 \prec 2 \prec 1 \prec 3 \prec 5 \prec \cdots$ is the module $X_{\mathfrak{sl}}(\mu)$, where $\mu_i = i$ for $i \in 2\mathbb{Z}_{>0} + 1$, $\mu_i = -i$ for $i \in 2\mathbb{Z}_{>0}$, and $\mu_1 = \pi$. A general criterion for a module to be a pseudo highest weight module is given in [GrP, Proposition 6.7 (ii)].

To conclude, we discuss briefly the case of non-integrable bounded weight $\mathfrak{sp}(\infty)$-modules. In this case, the modules $X_{\mathfrak{sp}}(\mu)$ are defined in a similar fashion. They are never integrable. Every non-integrable simple bounded highest weight $\mathfrak{sp}(\infty)$-module is isomorphic to $X_{\mathfrak{sp}}(\mu)$ for some μ. As a corollary, every simple bounded highest weight $\mathfrak{sp}(\infty)$-module is multiplicity free. An essential difference from the case of $\mathfrak{sl}(\infty)$ is that in the case of $\mathfrak{sp}(\infty)$ the support of a simple bounded highest weight module determines the module (see [GrP, Theorem 6.9 (iv)]).

8.3.4 Simple Highest Weight Modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ with Finite-Dimensional Weight Spaces

Let $\mathfrak{b}$ be an arbitrary splitting Borel subalgebra of $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. Then for any weight $\lambda \in \mathfrak{h}^*$, the Verma module $M_{\mathfrak{b}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ has a unique maximal submodule (the proof is the same as for finite-dimensional $\mathfrak{g}$), and hence, a unique simple quotient $L_{\mathfrak{b}}(\lambda)$. If $\mathfrak{p}$ is a splitting parabolic subalgebra and a weight $\lambda \in \mathfrak{h}^*$ is $\mathfrak{p}$-adapted, i.e., $\mathbb{C}_\lambda$ is a $\mathfrak{p}$-module, then the module $L_{\mathfrak{p}}(\lambda)$ is also well defined as the unique simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$.

If $\mathfrak{b}$ is a Dynkin Borel subalgebra, then it is easy to check that the Verma module $M_{\mathfrak{b}}(\lambda)$ has finite-dimensional weight spaces for any weight λ. Therefore all simple $\mathfrak{b}$ highest weight modules $L_{\mathfrak{b}}(\lambda)$ also have finite-dimensional weight spaces. We show now that the simple highest weight modules with respect to Dynkin Borel subalgebras account (up to isomorphism) for all simple highest weight modules that have finite-dimensional weight spaces, with a small exception.

Theorem 8.33 *Let $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. Let $L_{\mathfrak{b}}(\lambda)$ be a simple highest weight $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$ with respect to a splitting Borel subalgebra*

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. *Then the weight spaces of* $L_{\mathfrak{b}}(\lambda)$ *are finite dimensional if and only if either*

(1) there exists a Dynkin Borel subalgebra $\mathfrak{b}' = \mathfrak{h} \oplus \mathfrak{n}'$ *such that* $L_{\mathfrak{b}}(\lambda)$ *is a highest weight module with respect to* $\mathfrak{b}'$ *with the same highest weight* λ*, i.e.,* $L_{\mathfrak{b}}(\lambda) = L_{\mathfrak{b}'}(\lambda)$*,*
or
(2) $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ *and* $L_{\mathfrak{b}}(\lambda)$ *is isomorphic to the natural representation of* $\mathfrak{g}$*.*

This theorem is a slight modification of [DP1, Theorem 5 (iii)] in which statement *(2)* is missing. The proof of Theorem 8.33 presented here was kindly provided for us by I. Dimitrov.

First, we need the following lemma.

Lemma 8.34 *Let* $\mathfrak{g} = \mathfrak{gl}(n+2)$*, let* $\mathfrak{b} \subset \mathfrak{g}$ *be the Borel subalgebra consisting of upper triangular matrices and let* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n+1}) \in \mathfrak{h}^*$*. If* $\lambda_k \neq \lambda_0$ *and* $\lambda_k \neq \lambda_{n+1}$ *for all* $1 \leq k \leq n$*, then*

$$\dim L_{\mathfrak{b}}(\lambda)^{\lambda-(\varepsilon_0-\varepsilon_{n+1})} \geq n.$$

Proof Let $\mathrm{E}_{i,j}$ denote the matrix with 1 in the i, j-position and zeros elsewhere. For each $k = 1, \ldots, n$, we define elements X_k, Y_k of the universal enveloping algebra $U(\mathfrak{g})$ by

$$X_k := \mathrm{E}_{k,n+1}\mathrm{E}_{0,k}$$
$$Y_k := \mathrm{E}_{n+1,k}\mathrm{E}_{k,0}.$$

If v_λ is the highest weight vector of $L_{\mathfrak{b}}(\lambda)$, then $X_\ell Y_k v_\lambda = a_{\ell k} v_\lambda$, where

$$a_{\ell k} = \begin{cases} (\lambda_0 - \lambda_k)(\lambda_k - \lambda_{n+1}) & \ell = k \\ 0 & \ell > k \\ (\lambda_k - \lambda_\ell) & \ell < k. \end{cases} \tag{8.7}$$

Indeed, if $\ell = k$, then

$$\begin{aligned} X_k Y_k v_\lambda &= \mathrm{E}_{k,n+1}\mathrm{E}_{0,k}\mathrm{E}_{n+1,k}\mathrm{E}_{k,0}v_\lambda &&= \mathrm{E}_{k,n+1}\mathrm{E}_{n+1,k}\mathrm{E}_{0,k}\mathrm{E}_{k,0}v_\lambda \\ &= \mathrm{E}_{k,n+1}\mathrm{E}_{n+1,k}(\lambda_0 - \lambda_k)v_\lambda &&= (\lambda_0 - \lambda_k)(\lambda_k - \lambda_{n+1})v_\lambda. \end{aligned}$$

If $\ell > k$, then

$$\begin{aligned} X_\ell Y_k v_\lambda &= \mathrm{E}_{\ell,n+1}\mathrm{E}_{0,\ell}\mathrm{E}_{n+1,k}\mathrm{E}_{k,0}v_\lambda = \mathrm{E}_{\ell,n+1}\mathrm{E}_{n+1,k}\mathrm{E}_{0,\ell}\mathrm{E}_{k,0}v_\lambda \\ &= \mathrm{E}_{\ell,n+1}\mathrm{E}_{n+1,k}\mathrm{E}_{k,0}\mathrm{E}_{0,\ell}v_\lambda - \mathrm{E}_{\ell,n+1}\mathrm{E}_{n+1,k}\mathrm{E}_{k,\ell}v_\lambda = 0. \end{aligned}$$

The case $\ell < k$ is not needed here, and is left as an exercise for the reader. Formula (8.7) implies that if $\lambda_k \neq \lambda_0$ and $\lambda_k \neq \lambda_{n+1}$ for all $1 \leq k \leq n$, then the $(n \times n)$-matrix $A := (a_{\ell k})$ is invertible, which in turn implies that the vectors $v_k := Y_k v_\lambda$ are linearly independent vectors in $L_\mathfrak{b}(\lambda)^{\lambda-(\varepsilon_0-\varepsilon_{n+1})}$. Indeed, suppose that $x \in \mathbb{C}^n$ such that $\sum_{k=1}^n x_k Y_k v_\lambda = 0$. Then for each $\ell = 1, \ldots, n$, we have

$$0 = \sum_{k=1}^{n} x_k X_\ell Y_k v_\lambda = \sum_{k=1}^{n} x_k a_{\ell k} v_\lambda,$$

implying $\sum_{k=1}^n x_k a_{\ell k} = 0$. But then $Ax = 0$ and since A is invertible, we obtain $x = 0$. □

Proof of Theorem 8.33 One direction is almost trivial: the natural representations clearly have finite-dimensional weight spaces, and if $L_\mathfrak{b}(\lambda) \simeq L_{\mathfrak{b}'}(\lambda)$ for some Dynkin Borel subalgebra $\mathfrak{b}'$, then the Verma module $M_{\mathfrak{b}'}(\lambda)$ (and hence its simple quotient $L_{\mathfrak{b}'}(\lambda)$) has finite-dimensional weight spaces. Indeed, if $\mathfrak{b}'$ is a splitting Borel subalgebra, then any weight space of $M_{\mathfrak{b}'}(\lambda)$ coincides with a weight space of $M_{\mathfrak{b}'_n}(\lambda)$ for sufficiently large n (prove this!) and is therefore finite dimensional. Note, however, that for an arbitrary (splitting) Borel subalgebra, the weight spaces of a Verma module can be infinite dimensional (to the reader: construct such a Verma module for a perfect Borel subalgebra).

For the other direction, assume $L_\mathfrak{b}(\lambda)$ has finite-dimensional weight spaces. Let $\mathfrak{p} \supset \mathfrak{b}$ be the stabilizer of the highest weight space of $L_\mathfrak{b}(\lambda)$. Then $L_\mathfrak{b}(\lambda) \simeq L_\mathfrak{p}(\lambda)$ and $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ (see Sect. 5.2.3). Moreover, $L_\mathfrak{p}(\lambda) \simeq L_{\mathfrak{b}'}(\lambda)$ if and only if $\mathfrak{b}' \subset \mathfrak{p}$.

We first consider the case $\mathfrak{g} = \mathfrak{sl}(\infty)$. Then the parabolic subalgebra $\mathfrak{p}$ corresponds to a totally ordered partition $\{I_\alpha\}_{\alpha \in A}$ of the set $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$, i.e., a partition that is indexed by a totally ordered set A. On the other hand, a totally ordered partition $\{I_\alpha\}_{\alpha \in A}$ defines a partial order $\prec_\mathfrak{p}$ on $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$, where $\varepsilon_i \prec \varepsilon_j$ when $i \in I_\alpha$, $j \in I_\beta$ and $\alpha < \beta$.

We will use the notation λ_α, for $\alpha \in A$, to denote λ_i when $i \in I_\alpha$. This is well defined, since $\lambda_i = \lambda_j$ for any $i, j \in I_\alpha$. Note that if $\alpha < \beta$, and there does not exist $\gamma \in A$ such that $\alpha < \gamma < \beta$, then $\lambda_\alpha \neq \lambda_\beta$.

A Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}$ is contained in $\mathfrak{p}$ if and only if it corresponds to a total order on $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$ that is a refinement of the partial order $\prec_\mathfrak{p}$. Hence to prove the theorem, we need to show that $\prec_\mathfrak{p}$ admits a refinement that is isomorphic to $\mathbb{Z}_{\leq 0}$, $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}$. This is equivalent to the following two conditions on $\{I_\alpha\}_{\alpha \in A}$ being satisfied (prove!):

(A1) A is isomorphic as an ordered set to $\mathbb{Z}_{\leq 0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$ or a finite set.
(A2) Each of the parts I_α is finite, except for the possible exception of the ones indexed by the smallest and largest elements of A.

To complete the proof for $\mathfrak{g} = \mathfrak{sl}(\infty)$, we will show that the partition $\{I_\alpha\}_{\alpha \in A}$ satisfies conditions (A1) and (A2).

First, we prove (A2). Assume to the contrary, that $|I_\gamma| = \infty$ where γ is neither the largest, nor the smallest element of A. Then there exist $\alpha, \beta \in A$ where $\alpha < \gamma < \beta$ such that $\lambda_\gamma \neq \lambda_\alpha$ and $\lambda_\gamma \neq \lambda_\beta$. So for each $n \in \mathbb{Z}_{>0}$, we can fix $j_0 \in I_\alpha$, $j_1 \in I_\beta$ and $i_1, \ldots, i_n \in I_\gamma$, and then apply Lemma 8.34 to the subalgebra of $\mathfrak{g}$ corresponding to the indices $j_0, i_1, \ldots, i_n, j_1$ (which is isomorphic to $\mathfrak{sl}(n+2)$). This allows us to conclude that $\dim L_\mathfrak{b}(\lambda)^{\lambda-(\varepsilon_{j_0}-\varepsilon_{j_1})} \geq n$. Since n can be taken to be arbitrarily large, it follows that $\dim L_\mathfrak{b}(\lambda)^{\lambda-(\varepsilon_{j_0}-\varepsilon_{j_1})} = \infty$, which is a contradiction. This proves (A2).

Condition (A1) is equivalent to the following: for every $\alpha, \beta \in A$ with $\alpha < \beta$, the interval $(\alpha, \beta) := \{\eta \in A \mid \alpha < \eta < \beta\}$ is a finite set. So now assume to the contrary that some interval (α, β) is infinite, and consider λ_α, λ_β and λ_η for $\eta \in (\alpha, \beta)$.

If there are infinitely many $\eta \in (\alpha, \beta)$ for which $\lambda_\eta \neq \lambda_\alpha$ and $\lambda_\eta \neq \lambda_\beta$, then for each $n \in \mathbb{Z}_{>0}$, we can choose such $\eta_1, \ldots, \eta_n \in (\alpha, \beta)$ and apply Lemma 8.34 to the subalgebra of $\mathfrak{g}$ corresponding to the indices $j_0, i_1, \ldots, i_n, j_1$ for fixed $j_0 \in I_\alpha$, $j_1 \in I_\beta$ and $i_k \in I_{\eta_k}$. By the same argument as before, this yields a contradiction to the assumption that $\dim L_\mathfrak{b}(\lambda)^{\lambda-(\varepsilon_{j_0}-\varepsilon_{j_1})} < \infty$.

Now we may assume, without loss of generality, that there are infinitely many $\eta \in (\alpha, \beta)$ for which $\lambda_\eta = \lambda_\beta$.

Suppose first that there exists an infinite sequence

$$\eta_1 < \eta_2 < \eta_3 < \cdots, \text{ with } \eta_k \in (\alpha, \beta) \text{ and } \lambda_{\eta_k} = \lambda_\beta. \tag{8.8}$$

Since $\eta_k < \eta_{k+1}$ and $\lambda_{\eta_k} = \lambda_{\eta_{k+1}} = \lambda_\beta$, there exists $\eta_{k'}$ such that $\eta_k < \eta_k' < \eta_{k+1}$ and $\lambda_{\eta_k'} \neq \lambda_\beta$. Taking $j_0 \in I_{\eta_1}$, $j_1 \in I_\beta$ and $i_1, i_2, i_3, \ldots, i_n$ respectively in $I_{\eta_1'}, I_{\eta_2'}, \ldots, I_{\eta_n'}$ we can apply Lemma 8.34 to the subalgebra of $\mathfrak{g}$ corresponding to the indices $j_0, i_1, i_2, i_3, \ldots, i_n, j_1$ to reach a contradiction.

Suppose now that such a sequence (8.8) does not exist. Then we can choose $\beta' \in (\alpha, \beta)$ such that $\lambda_{\beta'} \neq \lambda_\beta$ and (α, β') contains infinitely many η with $\lambda_\eta = \lambda_\beta$. If $\lambda_\alpha \neq \lambda_\beta$, then we are in an earlier case: infinitely many $\eta \in (\alpha, \beta')$ for which $\lambda_\eta \neq \lambda_\alpha$ and $\lambda_\eta \neq \lambda_{\beta'}$, and so the claim follows.

If $\lambda_\alpha = \lambda_\beta$, then we have two cases to consider, namely there exists an infinite sequence $\eta_1 > \eta_2 > \eta_3 > \cdots$ with $\eta_k \in (\alpha, \beta')$ and $\lambda_{\eta_k} = \lambda_\alpha$, or there does not. The first case is handled by arguing as above, while in the second case we can choose $\alpha' \in (\alpha, \beta')$ such that $\lambda_{\alpha'} \neq \lambda_\alpha$ and (α', β') contains infinitely many η with $\lambda_\eta = \lambda_\alpha$. Then since $\lambda_\alpha \neq \lambda_{\alpha'}, \lambda_{\beta'}$, this reduces to a previous case, and hence we are done. This completes the proof for $\mathfrak{g} = \mathfrak{sl}(\infty)$.

Now we prove the case of $\mathfrak{g} = \mathfrak{o}(\infty)$ with splitting Borel subalgebra $\mathfrak{b}$ of type B. The cases $\mathfrak{g} = \mathfrak{o}(\infty)$ of type D and $\mathfrak{g} = \mathfrak{sp}(\infty)$ are treated similarly, and will not be discussed here. As in the case of $\mathfrak{g} = \mathfrak{sl}(\infty)$, we consider a splitting parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$ (see Sect. 5.2.4). Here $\mathfrak{p}$ corresponds to a totally ordered partition $\{I_\alpha\}_{\alpha \in A}$ of the set $\{\pm\varepsilon_1, \pm\varepsilon_2, \ldots, 0\}$ that is compatible with multiplication by -1. We denote by I_0 the part containing 0, and can assume $A = -A$ where $I_{-\alpha} = -I_\alpha$. Note that $\lambda_0 = 0$ since $0 \in I_0$.

Let $A^+ = \{\alpha \in A \mid \alpha > 0\}$. For any choice of signs $\sigma_\alpha \in \{\pm 1\}$, we obtain a subset A' of A:

$$A' := \{\sigma_\alpha \alpha \mid \alpha \in A^+\}.$$

Since the subalgebra of $\mathfrak{g}$ corresponding to $\cup_{\alpha \in A'} I_\alpha$ is isomorphic to $\mathfrak{sl}(\infty)$ or $\mathfrak{sl}(n)$, we have by the previous case that A' must satisfy (A1) and (A2).

First, we claim that the ordered set A^+ is either finite or isomorphic to $\mathbb{Z}_{\geq 0}$. Indeed, if A^+ were isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_{\leq 0}$, then the set $A'' := (A^+ \setminus \{\beta\}) \cup \{-\beta\}$, obtained from A^+ by changing the sign of some element $\beta \in A^+$, would not satisfy (A1) as $-\beta < 0 < \alpha$ for every $\alpha \in A^+ \setminus \{\beta\}$. This is a contradiction, so A^+ is either finite or isomorphic to $\mathbb{Z}_{\geq 0}$.

Next, we claim that I_0 is finite unless $L_{\mathfrak{b}}(\lambda)$ is a trivial module or is isomorphic to the natural representation of $\mathfrak{g}$. Suppose that I_0 is infinite. Then there cannot be two indices $j_1 \neq j_2$ with $\varepsilon_{j_1} > 0$, $\varepsilon_{j_2} > 0$ and $\lambda_{j_1} \neq 0$, $\lambda_{j_2} \neq 0$, because we could consider the $\mathfrak{sl}(\infty)$-subalgebra corresponding to $\{\varepsilon_1, -\varepsilon_2\} \cup I_0 \setminus \{0\}$ and apply Lemma 8.34 to reach a contradiction. This implies (without loss of generality) that either

(a) A^+ is empty, or
(b) A^+ consists of a single element α and $I_\alpha = \{\varepsilon_1\}$, or
(c) $A^+ = \{\alpha, \beta\}$, $\alpha > \beta$, $\lambda_\alpha = 0$, $\varepsilon_1 \in I_\alpha$ and $I_\beta = \{\varepsilon_2\}$.

For case (a), we get $\lambda_i = 0$ for all i, and so $L_{\mathfrak{b}}(\lambda)$ is a trivial module. In case (b), the highest weight of $L_{\mathfrak{b}}(\lambda)$ is $\lambda_1 \varepsilon_1$ with respect to the Borel subalgebra corresponding to the order $\varepsilon_1 > \varepsilon_2 > \cdots > 0$. A direct calculation analogous to the proof of Lemma 8.34 shows that

$$\dim L_{\mathfrak{b}}(\lambda)^{\lambda - 2\varepsilon_1} = \infty$$

unless $\lambda_1 = 1$, and in which case $L_{\mathfrak{b}}(\lambda)$ is isomorphic to the natural representation of $\mathfrak{g}$. (This calculation is exactly the same as for Lemma 8.34, and is left as an exercise for the reader. One should prove separately for B_n, C_n, D_n that the dimension of the weight space $\lambda - 2\varepsilon_1$ is greater than or equal to $n, n, n-1$, respectively.) To deal with case (c), consider the Borel subalgebra $\mathfrak{b}'$ corresponding to the order $\varepsilon_1 > \varepsilon_2 > \cdots > 0$. Now $\lambda = \lambda_2 \varepsilon_2$, and by applying (b) for the $\mathfrak{o}(\infty)$-subalgebra $\tilde{\mathfrak{g}}$ corresponding to $\{\varepsilon_2\} \cup I_0$, we obtain $\lambda = \varepsilon_2$. Let v_0 denote the highest weight vector with respect to $\mathfrak{b}'$. Then for a non-zero vector $x_{-(\varepsilon_1 - \varepsilon_2)} \in \mathfrak{g}_{-(\varepsilon_1 - \varepsilon_2)}$, the vector $v_1 := x_{-(\varepsilon_1 - \varepsilon_2)} v_0$ is a singular vector for $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{b}' \cap \tilde{\mathfrak{g}}$ because $\varepsilon_1 - \varepsilon_2$ is a simple root of $\mathfrak{g}$ that is not a root of $\tilde{\mathfrak{g}}$. Since the weight of v_1 with respect to $\mathfrak{h} \cap \tilde{\mathfrak{g}}$ is $2\varepsilon_2$, applying (b) for $\tilde{\mathfrak{g}}$ yields a contradiction, completing the proof of this claim.

Finally, we claim that if I_γ is infinite for some $\gamma \in A^+$, then γ is the largest element of A^+. Indeed, suppose not. Let $\alpha \in A^+$ such that $\alpha > \gamma$ and $\lambda_\alpha \neq \lambda_\gamma$. If $\lambda_\gamma = 0$, then we can take $\beta \in A^+$ with $\beta < \gamma$, $\lambda_\beta \neq 0$, and reach a contradiction to

Lemma 8.34 using the set $\{\beta\} \cup I_\gamma \cup \{\alpha\}$. While if $\lambda_\gamma \neq 0$, then by choosing $\beta \in I_\gamma$ we reach a contradiction by using the set $\{-\beta\} \cup (I_\gamma \setminus \{\beta\}) \cup \{\alpha\}$.

Therefore if I_0 is a finite set, then the partial order $\prec_{\mathfrak{p}}$ can be refined to a total order on $\{\pm\varepsilon_1, \pm\varepsilon_2, \ldots, 0\}$ producing a Dynkin Borel subalgebra $\mathfrak{b}' \subset \mathfrak{p}$ with $L_{\mathfrak{b}}(\lambda) = L_{\mathfrak{p}}(\lambda) = L_{\mathfrak{b}'}(\lambda)$. This completes the proof of Theorem 8.33. □

Theorem 8.33 implies an explicit description of the simple integrable highest weight modules that have finite-dimensional weight spaces. Indeed, as every such module has a highest weight with respect to a Dynkin Borel subalgebra, it follows that every such module is isomorphic to $L_{\mathfrak{b}}(\lambda)$, where $\mathfrak{b}$ is a Dynkin Borel subalgebra and λ is an integral dominant weight, i.e., $\frac{(\lambda,\alpha)}{2(\alpha,\alpha)} \in \mathbb{Z}_{\geq 0}$ for all simple roots α of $\mathfrak{b}$.

Explicitly this means that $\lambda = \sum \lambda_i \varepsilon_i$ satisfies the following conditions:

- if $\mathfrak{g} = \mathfrak{sl}(\infty)$, then $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$ whenever $\varepsilon_i - \varepsilon_j$ is a simple root of $\mathfrak{b}$;
- if $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{b}$ has simple roots $\{\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots\}$, then $\lambda_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, and $\lambda_j - \lambda_j \in \mathbb{Z}_{\geq 0}$ whenever $i < j$;
- if $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{b}$ has simple roots $\{\varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots\}$, then $\lambda_1 \in \frac{1}{2}\mathbb{Z}$, $\lambda_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ for $i = 2, 3, \ldots$, $|\lambda_1| \leq \lambda_2$ and $\lambda_j - \lambda_j \in \mathbb{Z}_{\geq 0}$ whenever $i < j$;
- if $\mathfrak{g} = \mathfrak{sp}(\infty)$ and $\mathfrak{b}$ has simple roots $\{2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots\}$, then $\lambda_i \in \mathbb{Z}_{\geq 0}$, and $\lambda_j - \lambda_j \in \mathbb{Z}_{\geq 0}$ whenever $i < j$.

The reader can verify that any Dynkin Borel subalgebra of $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ is conjugate by the group $\operatorname{Aut}\mathfrak{g}$ of automorphisms of $\mathfrak{g}$ to the respective Borel subalgebra given in the above list. Now this list is complete.

For a fixed Dynkin Borel subalgebra $\mathfrak{b}$, the module $L_{\mathfrak{b}}(\lambda)$ is isomorphic to a module $L_{\mathfrak{b}}(\mu)$ if and only if $\lambda = \mu$. Of course if two Dynkin Borel subalgebras $\mathfrak{b}$, $\mathfrak{b}'$ are conjugate, then we can have isomorphisms between two modules $L_{\mathfrak{b}}(\lambda)$ and $L_{\mathfrak{b}'}(\mu)$. For $\mathfrak{g} = \mathfrak{sl}(\infty)$, not all Dynkin Borel subalgebras are conjugate under $\operatorname{Aut}\mathfrak{g}$, and in this case, an isomorphism of simple integrable modules $L_{\mathfrak{b}}(\lambda) \simeq L_{\mathfrak{b}'}(\mu)$ for non-conjugate Dynkin Borel subalgebras containing a fixed splitting Cartan subalgebra exists if and only if $\lambda = \mu = 0$ (prove!).

8.4 Categories of Modules over Root-Reductive Lie Algebras

In this final section, we mention some results on categories of modules over a root-reductive Lie algebra $\mathfrak{g}$.

8.4.1 Categories of Integrable Modules

Consider the category $\mathrm{Int}_{\mathfrak{g}}$ of all integrable $\mathfrak{g}$-modules. In contrast with the case of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, for which the category of integrable

modules is semisimple, the category $\mathrm{Int}_{\mathfrak{g}}$ is a very rich category and is far from being semisimple. Here are some interesting subcategories of $\mathrm{Int}_{\mathfrak{g}}$:

(1) The category $\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$ of integrable $\mathfrak{h}$-weight $\mathfrak{g}$-modules with finite-dimensional weight spaces.

A theorem generalizing Weyl's semisimplicity theorem claims that this category is semisimple [PS6, Theorem 3.7]. Clearly there are two types of simple objects of $\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$: simple modules that have a highest weight with respect to some splitting Borel subalgebra $\mathfrak{b}$, and simple modules that admit no highest weight for any splitting Borel subalgebra. In fact, the former type consists of modules that have a highest weight with respect to some Dynkin Borel subalgebra, with a small exception (see Theorem 8.33 (2)). Simple integrable bounded modules that have no highest weight (see Sect. 8.3.3) belong to the latter type, but there exist also unbounded simple integrable modules without a highest weight and yet with finite-dimensional weight spaces.

(2) The category of tensor modules $\mathbb{T}_{\mathfrak{g}}$ (see Chap. 7).

(3) The category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ of all integrable modules of finite Loewy length that have the property that their algebraic duals are also integrable of finite Loewy length (recall the definition of Loewy length from Sect. 7.1).

It is a remarkable result that the simple objects of the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are the same as of the category $\mathbb{T}_{\mathfrak{g}}$ [PS6, Theorem 6.4]; in fact, $\mathbb{T}_{\mathfrak{g}}$ is a full subcategory of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. This implies that if $M \in \mathbb{T}_{\mathfrak{g}}$ is any tensor module, then any finite iterated dual $M^{*\cdots *}$ has finite Loewy length, and its simple subquotients are of the form $V_{\lambda,\mu}$ (where λ and μ run over some set of partitions dependent on M). Of course, unless M is a trivial module, $M^{*\cdots *}$ is not itself an object of $\mathbb{T}_{\mathfrak{g}}$ as it has uncountable length. For instance, when $\mathfrak{g} = \mathfrak{sl}(\infty)$ the reader can try to verify that the module $V^*_{(1),(1)} = \mathfrak{g}^*$ has socle filtration of the following form:

$2^{\aleph_0} V_{\emptyset,\emptyset}$
$2^{\aleph_0} V_{(1),\emptyset} \oplus 2^{\aleph_0} V_{\emptyset,(1)} \oplus 2^{\aleph_0} V_{\emptyset,\emptyset}$
$V_{(1),(1)}$

Recall that $V_{(1),\emptyset}$ and $V_{\emptyset,(1)}$ are the natural and conatural $\mathfrak{sl}(\infty)$-modules, respectively, and that $V_{\emptyset,\emptyset}$ is a one-dimensional trivial module. This example is discussed in [PS6, Example 6.2].

As a further exercise, the reader can determine the socle filtrations of the modules $\mathfrak{g}^*$ for $\mathfrak{g} = \mathfrak{o}(\infty)$ and $\mathfrak{g} = \mathfrak{sp}(\infty)$.

Extensive results on the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are presented in [PS6]. In addition, it was noticed in [CP1, Section 3] that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is an infinite ordered tensor category (this is a notion that slightly generalizes the notion of ordered tensor category introduced in Sect. 7.3), which led to the description of arbitrary injective objects in $\mathbb{T}_{\mathfrak{g}}$ [CP1].

8.4.2 Categories Containing Non-Integrable Modules

Next we mention some categories of not necessarily integrable $\mathfrak{g}$-modules. In the finite-dimensional case, the BGG category $\mathcal{O}$ is a prominent such example. When trying to extend the definition of this category to root-reductive Lie algebras, we face an obvious problem: such a definition must depend heavily on a choice of a Borel subalgebra. In the work of T. Nampaisarn [Nam], this choice is restricted to a Dynkin Borel subalgebra.

More precisely, let $\mathfrak{b} = \mathfrak{h} \supset \mathfrak{n}$ be a Dynkin Borel subalgebra of a root-reductive Lie algebra $\mathfrak{g}$. T. Nampaisarn studies the category $\overline{\mathcal{O}}_{\mathfrak{b}}$ of $\mathfrak{g}$-modules M satisfying the following two conditions:

- M is an $\mathfrak{h}$-weight module with finite-dimensional weight spaces;
- the action of the Lie algebra $\mathfrak{n}$ on M is locally finite.

The fact that $\mathfrak{b}$ is a Dynkin Borel subalgebra implies that all Verma modules with respect to $\mathfrak{b}$ have finite-dimensional weight spaces, and hence are objects of $\overline{\mathcal{O}}_{\mathfrak{b}}$. In the paper [Nam], one can find a Jordan–Hoelder type theorem for $\overline{\mathcal{O}}_{\mathfrak{b}}$ and a description of the blocks of $\overline{\mathcal{O}}_{\mathfrak{b}}$. A modification of the category $\overline{\mathcal{O}}_{\mathfrak{b}}$ has been introduced and studied in [CouP].

Note that the category $\mathbb{T}_{\mathfrak{g}}$ is not a subcategory of the category $\overline{\mathcal{O}}_{\mathfrak{b}}$ (for any choice of Dynkin Borel subalgebra) as most of its simple modules have infinite-dimensional weight spaces. It is an interesting problem to find an "$\mathcal{O}$-type category" whose simple integrable modules would be exactly the simple tensor modules. This problem is solved in the work of [PS9], where the category $\mathcal{OLA}$ is introduced for $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. Such a category consists of $\mathfrak{g}$-modules M satisfying the following conditions with respect to a perfect Borel subalgebra $\mathfrak{b} = \mathfrak{h} \supset \mathfrak{n}$ (for $\mathfrak{g} = \mathfrak{sl}(\infty)$, one can consider the Borel subalgebra from Example 5.4, and for $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, the reader can find a definition of perfect Borel subalgebra in [PS9]):

- M is a $\mathfrak{h}$-weight module;
- every one-dimensional Lie subalgebra of $\mathfrak{n}$ acts locally finitely on M;
- $\mathrm{Ann}_{\mathfrak{g}} m$ is finite corank for each $m \in M$.

The category $\mathcal{OLA}$ has remarkable properties which differ essentially from the properties of the category $\overline{\mathcal{O}}_{\mathfrak{b}}$. For instance, every finitely generated module has finite length (to the reader: prove that this is false in $\overline{\mathcal{O}}_{\mathfrak{b}}$), and moreover, $\mathcal{OLA}$ is a highest weight category in terms of the definition of [CPS].

8.5 Closing Remarks

We conclude this chapter with a few remarks.

1. We should point out that Yu. Drozd, S. Futorny, S. Ovsienko, V. Mazorchuk, et al. have developed a theory for a class of weight modules called Gelfand–

Tsetlin modules. This theory generalizes the finite-dimensional modules of finite-dimensional Lie algebras in a direction that is beyond the scope of this book. A short list of references on this topic is [DFO, FO, Maz, Web].

2. In the setting of root-reductive Lie algebras, classifying simple weight modules with finite-dimensional weight spaces is still an open problem.
3. For classical simple root-reductive Lie superalgebras, including $\mathfrak{sl}(\infty|\infty)$, L. Calixto and the first author have classified simple integrable bounded weight modules [CalP].

Chapter 9
Generalized Harish-Chandra Modules

In this chapter we study a class of $\mathfrak{g}$-modules that generalize weight modules with finite-dimensional weight spaces by replacing the Cartan subalgebra $\mathfrak{h}$ with an arbitrary subalgebra $\mathfrak{k}$. We call these modules generalized Harish-Chandra modules.

9.1 Introduction

Let $\mathfrak{g}$ be a locally finite Lie algebra, let M be a $\mathfrak{g}$-module, and let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra. By definition, M is a $(\mathfrak{g}, \mathfrak{k})$-*module* if $\mathfrak{k} \subset \mathfrak{g}[M]$, where $\mathfrak{g}[M]$ denotes the Fernando–Kac subalgebra of $\mathfrak{g}$ associated to M (see Theorem 8.1). In other words a $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{k})$-module if and only if M is integrable over $\mathfrak{k}$. Consequently, a $\mathfrak{g}$-module M is integrable if and only if M is a $(\mathfrak{g}, \mathfrak{g})$-module. Following [PZ4], we say that M is a $(\mathfrak{g}, \mathfrak{k})$-module of *finite type* (or an *admissible* $(\mathfrak{g}, \mathfrak{k})$-module) if the following two conditions hold:

(f1) every finitely generated $\mathfrak{k}$-submodule M' of M has finite length as a $\mathfrak{k}$-module;
(f2) for every fixed simple integrable $\mathfrak{k}$-module L, the multiplicity of L as a subquotient of M' is bounded when M' runs over all finitely generated $\mathfrak{k}$-submodules of M.

We say that M is a *generalized Harish-Chandra module* if M is a $(\mathfrak{g}, \mathfrak{g}[M])$-module of finite type. Any $(\mathfrak{g}, \mathfrak{k})$-module of finite type is a generalized Harish-Chandra module since if M satisfies conditions (f1) and (f2) for the pair $(\mathfrak{g}, \mathfrak{k})$, then it automatically satisfies the same conditions for the pair $(\mathfrak{g}, \mathfrak{g}[M])$.

The study of generalized Harish-Chandra modules is still under development and stands on the shoulders of the theory of Harish-Chandra modules, a most celebrated area of representation theory. The books [Vo2] and [KnVo] can give a rough idea of the depth of the theory of Harish-Chandra modules. In this chapter, we briefly describe some landmark results concerning generalized Harish-Chandra modules.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_9

An early introduction to the relevant problem setting can be found in [PZ1]. This is probably the most technically challenging chapter.

In order to illustrate the notion of a generalized Harish-Chandra module, let's consider the simplest case $\mathfrak{g} = \mathfrak{sl}(2)$. It is well known that up to conjugation there are five possibilities for a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Let's list them: first, $\mathfrak{k} = \mathfrak{g}$; second, $\mathfrak{k} = \mathfrak{b}$, a Borel subalgebra of $\mathfrak{g}$; third, $\mathfrak{k} = \mathfrak{n}$ where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$; fourth, $\mathfrak{k} = \mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$; and finally $\mathfrak{k} = \{0\}$. Any simple $\mathfrak{g}$-module is a $(\mathfrak{g}, \mathfrak{k})$-module for some $\mathfrak{k}$ as above, and most of the corresponding $(\mathfrak{g}, \mathfrak{k})$-modules have a classical name. For $\mathfrak{k} = \mathfrak{g}$, a simple $(\mathfrak{g}, \mathfrak{k})$-module is just a simple finite-dimensional representation of $\mathfrak{sl}(2)$. For $\mathfrak{k} = \mathfrak{b}$, a simple $(\mathfrak{g}, \mathfrak{k})$-module is a simple module with a highest weight with respect to $\mathfrak{b}$. For $\mathfrak{k} = \mathfrak{n}$, the simple $(\mathfrak{g}, \mathfrak{k})$-modules that are not $(\mathfrak{g}, \mathfrak{b})$-modules are the well-known nondegenerate simple Whittaker modules. For $\mathfrak{k} = \mathfrak{h}$, the simple $(\mathfrak{g}, \mathfrak{h})$-modules are the Harish-Chandra modules of $\mathfrak{sl}(2)$ (see the definition below). In this case, simple Harish-Chandra modules are nothing but simple weight modules, and they do not have to have a highest weight (see Sect. 8.2.3). Finally, simple $(\mathfrak{g}, \mathfrak{k})$-modules exist also for $\mathfrak{k} = \{0\}$. A construction of such modules was given in 1974 by D. Arnal and G. Pinczon [AP], and later such modules appeared again in R. Block's description of all simple $\mathfrak{sl}(2)$-modules [Bl].

To summarize, the following holds for $\mathfrak{g} = \mathfrak{sl}(2)$. If $\mathfrak{k} = \{0\}$ or $\mathfrak{k} = \mathfrak{n}$, then there are no infinite-dimensional simple generalized Harish-Chandra modules M with $\mathfrak{k} = \mathfrak{g}[M]$ as the condition (f2) is violated. For $\mathfrak{k} = \{0\}$ this is clear, and for $\mathfrak{k} = \mathfrak{n}$ it is a consequence of the structure of Whittaker modules [Ko3]. On the other hand, if $\mathfrak{k} = \mathfrak{b}$ or $\mathfrak{k} = \mathfrak{h}$, then there exist infinite-dimensional simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type. This follows from the description of all simple weight $\mathfrak{sl}(2)$-modules, as we know from Sect. 8.2.3. Moreover, if $\mathfrak{k} = \mathfrak{g}$, $\mathfrak{k} = \mathfrak{b}$ or $\mathfrak{k} = \mathfrak{h}$, then every simple $(\mathfrak{g}, \mathfrak{k})$-module is a generalized Harish-Chandra module, as every simple $(\mathfrak{g}, \mathfrak{h})$-module is a weight module with one-dimensional weight spaces.

Next, we would like to point out three special classes of generalized Harish-Chandra modules for more general Lie algebras $\mathfrak{g}$.

First, note that a weight module with finite-dimensional weight spaces is a generalized Harish-Chandra module. Indeed, a weight module $M = \bigoplus_{\nu \in \mathfrak{h}^*} M^\nu$ with $\dim M^\nu < \infty$ is a $(\mathfrak{g}, \mathfrak{h})$-module of finite type, and hence a generalized Harish-Chandra module. Therefore Chap. 8 can be considered as a specific chapter in the theory of generalized Harish-Chandra modules.

Second, one of the most classical cases of generalized Harish-Chandra modules is the case when $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra and $\mathfrak{k}$ is a symmetric subalgebra, i.e., $\mathfrak{k}$ coincides with the fixed points of an involution $i : \mathfrak{g} \to \mathfrak{g}$. Such a pair $(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair. Under the additional assumption that $\mathfrak{k}$ acts semisimply, the corresponding generalized Harish-Chandra modules are called *Harish-Chandra modules*. (Note that this definition is more restrictive than the definition given in [Dix].)

It is a celebrated theorem of Harish-Chandra that if $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, then any simple $(\mathfrak{g}, \mathfrak{k})$-module M automatically has finite type over $\mathfrak{k}$ (in other words, M is admissible over $\mathfrak{k}$).

A pair $(\mathfrak{g}, \mathfrak{k})$ is *reducible* if $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}''$, $\mathfrak{k}' \subset \mathfrak{g}'$, $\mathfrak{k}'' \subset \mathfrak{g}''$. Here is a list of all irreducible symmetric pairs of classical Lie algebras (where $n \geq 2$):

- $(\mathfrak{sl}(n), \mathfrak{sl}(k) \oplus \mathfrak{sl}(n-k) \oplus \mathbb{C})$, $0 < k < n$,
- $(\mathfrak{o}(n), \mathfrak{o}(k) \oplus \mathfrak{o}(n-k))$, $0 < k < n$, $n \geq 3$,
- $(\mathfrak{sp}(2n), \mathfrak{sp}(2k) \oplus \mathfrak{sp}(2(n-k))$, $0 < k < n$,
- $(\mathfrak{sl}(n), \mathfrak{o}(n))$,
- $(\mathfrak{sl}(2n), \mathfrak{sp}(2n))$,
- $(\mathfrak{o}(2n), \mathfrak{gl}(n))$,
- $(\mathfrak{sp}(2n), \mathfrak{gl}(n))$,
- $(\mathfrak{g} \oplus \mathfrak{g}$, diagonal $\mathfrak{g})$, $\mathfrak{g}$ a classical simple Lie algebra.

This classification is essentially equivalent to E. Cartan's famous classification of symmetric spaces from 1926. Cartan's classification covers also the case of exceptional Lie groups whose discussion we omit here. We refer the reader instead to [Hel] and [Kn]. See also [K1] for a classification of finite order automorphisms of semisimple finite-dimensional Lie algebras.

A third type of generalized Harish-Chandra modules to which we pay special attention is the case when $\mathfrak{g}$ is semisimple and $\mathfrak{k} \simeq \mathfrak{sl}(2)$ (see Sect. 9.2.4). A classification of $\mathfrak{sl}(2)$-subalgebras of semisimple Lie algebras has been given by E. Dynkin in [Dyn] based in part on the earlier work of A. Malcev [Mal].

Our main tool for constructing generalized Harish-Chandra modules in this chapter will be the Zuckerman functor. Certainly, there are other methods of constructing generalized Harish-Chandra modules which remain beyond the scope of this book, see for example [BB, E, KnVo].

Here is the definition of the Zuckerman functor. Let $\mathfrak{g}$ be any Lie algebra and let $\mathfrak{k}' \subset \mathfrak{g}$ be a finite-dimensional subalgebra that acts locally finitely and semisimply on $\mathfrak{g}$. For instance, if $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ is locally reductive and $\mathfrak{k}' \subset \mathfrak{g}_n$ is a reductive in $\mathfrak{g}_n$ subalgebra for some n, then the above condition is satisfied.

Denote by $C(\mathfrak{g}, \mathfrak{k}')$ the category of all $(\mathfrak{g}, \mathfrak{k}')$-modules that are semisimple over $\mathfrak{k}'$. For any reductive in $\mathfrak{k}'$ subalgebra $\mathfrak{m}' \subset \mathfrak{k}'$ ($\mathfrak{m}'$ may equal zero), we consider the left exact functor

$$\Gamma_{\mathfrak{k}',\mathfrak{m}'} : C(\mathfrak{g}, \mathfrak{m}') \to C(\mathfrak{g}, \mathfrak{k}') \tag{9.1}$$

$$M \mapsto \Gamma_{\mathfrak{k}',\mathfrak{m}'}(M) := \sum_{X \subset M, X \in C(\mathfrak{g},\mathfrak{k}')} X \ . \tag{9.2}$$

In words, $\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M)$ is the unique maximal $\mathfrak{k}'$-semisimple and $\mathfrak{k}'$-integrable $\mathfrak{g}$-submodule of M. Note that $\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M)$ is a submodule of the $\mathfrak{g}$-module $M^{(\mathfrak{k})}$ from Theorem 8.2.

Moreover, the category $C(\mathfrak{g}, \mathfrak{m}')$ has enough injective objects [Z, Lemma 2.2], and hence one can introduce the right derived functor

$$R^{\cdot}\Gamma_{\mathfrak{k}',\mathfrak{m}'} = \bigoplus_{j \geq 0} R^j \Gamma_{\mathfrak{k}',\mathfrak{m}'}. \tag{9.3}$$

This functor is known as the *Zuckerman functor* ($R^j\Gamma_{\mathfrak{k}',\mathfrak{m}'}$ is the jth-*Zuckerman functor*), and is defined up to isomorphism as we do not provide a canonical injective resolution for its computation. Of course, $R^0\Gamma_{\mathfrak{k}',\mathfrak{m}'} = \Gamma_{\mathfrak{k}',\mathfrak{m}'}$.

A well-known property of the Zuckerman functor which we use below is that if the center $Z_{U(\mathfrak{g})}$ of the universal enveloping algebra $U(\mathfrak{g})$ acts via a fixed character on M, then $Z_{U(\mathfrak{g})}$ acts via the same character on $R^\cdot\Gamma_{\mathfrak{k}',\mathfrak{m}'}(M)$.

9.2 The Case where $\mathfrak{g}$ is Finite Dimensional

Below, we review some central results in the algebraic theory of generalized Harish-Chandra modules due to Zuckerman, Serganova and the first author. Parts of this work have been inspired by the groundbreaking work of Vogan [Vo2]. Throughout the entire chapter, we have adapted key results from the papers [PZ2, PZ4, PSZ2], and some parts of the text follow closely the original sources.

In this part of the chapter, we assume $\mathfrak{g}$ to be finite dimensional and semisimple and $\mathfrak{k}$ to be an arbitrary reductive in $\mathfrak{g}$ algebraic subalgebra. Furthermore, by a $(\mathfrak{g},\mathfrak{k})$-module, we understand a $(\mathfrak{g},\mathfrak{k})$-module on which $\mathfrak{k}$ acts semisimply.

9.2.1 *Construction of* $(\mathfrak{g},\mathfrak{k})$*-Modules*

Our starting question is: given a semisimple (finite-dimensional) Lie algebra $\mathfrak{g}$ and a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$, when does there exist an infinite-dimensional simple $(\mathfrak{g},\mathfrak{k})$-module of finite type? The answer is given in Theorem 9.7.

We will present a construction of $(\mathfrak{g},\mathfrak{k})$-modules, but first we need to fix quite a bit of notation.

A finite *multiset* is a function $f_\cdot$ from a finite set D into $\mathbb{Z}_{\geq 0}$. A *submultiset* of $f_\cdot$ is a multiset $f'_\cdot$ defined on the same domain D such that $f'_d \leq f_d$ for any $d \in D$. For any finite multiset $f_\cdot$, defined on an additive monoid D, we can set $\rho_{f_\cdot} := \frac{1}{2}\sum_{d\in D} f_d d$. If $\mathfrak{l}$ is a Lie algebra and M is a weight $\mathfrak{l}$-module with $\dim M < \infty$, then M determines the finite multiset $\mathrm{ch}_\mathfrak{l} M$, which is by definition the function $\omega \mapsto \dim M^\omega$ defined on $\mathrm{supp}_\mathfrak{l} M$.

Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{h}$. Note $\mathfrak{g}$ is a $\mathfrak{t}$-weight module as $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Note also that the $\mathbb{R}$-span of the roots Δ of $\mathfrak{h}$ in $\mathfrak{g}$ fixes a real structure on $\mathfrak{h}^*$, whose projection onto $\mathfrak{t}^*$ is a well-defined real structure on $\mathfrak{t}^*$. In what follows, we will denote by $\mathrm{Re}\,\lambda$ the real part of an element $\lambda \in \mathfrak{t}^*$. We also fix a Borel subalgebra $\mathfrak{b}_\mathfrak{k} \subset \mathfrak{k}$ with $\mathfrak{b}_\mathfrak{k} \supset \mathfrak{t}$. Then $\mathfrak{b}_\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{n}_\mathfrak{k}$, where $\mathfrak{n}_\mathfrak{k}$ is the nilradical of $\mathfrak{b}_\mathfrak{k}$. By $W_\mathfrak{k}$ we denote the Weyl group of $\mathfrak{k}$. Let $\rho \in \mathfrak{t}^*$ be the half sum of roots of the Borel subalgebra $\mathfrak{b}_\mathfrak{k}$ of $\mathfrak{k}$.

Let $\langle\,,\,\rangle$ denote the unique $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$ such that $\langle\alpha,\alpha\rangle = 2$ for any long root of a simple component of $\mathfrak{g}$. The form $\langle\,,\,\rangle$ enables us

to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then $\mathfrak{h}$ is identified with $\mathfrak{h}^*$, and $\mathfrak{k}$ is identified with $\mathfrak{k}^*$. We will sometimes consider $\langle\ ,\ \rangle$ as a form on $\mathfrak{g}$. The superscript $\perp$ indicates orthogonal space. Note that there is a canonical $\mathfrak{k}$-module decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$. We also set $\|\kappa\|^2 := \langle \kappa, \kappa \rangle$ for any $\kappa \in \mathfrak{h}^*$.

To any $\lambda \in \mathfrak{t}^*$ we associate the following parabolic subalgebra $\mathfrak{p}_\lambda$ of $\mathfrak{g}$:

$$\mathfrak{p}_\lambda = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_\lambda} \mathfrak{g}^\alpha \right),$$

where $\Delta_\lambda := \{\alpha \in \Delta \mid \langle \operatorname{Re} \lambda, \alpha \rangle \geq 0\}$. We call $\mathfrak{p}_\lambda$ a *compatible parabolic subalgebra*. By $\mathfrak{m}_\lambda$ and $\mathfrak{n}_\lambda$ we denote, respectively, the reductive part of $\mathfrak{p}_\lambda$ (containing $\mathfrak{h}$) and the nilradical of $\mathfrak{p}_\lambda$. In particular, $\mathfrak{p}_\lambda = \mathfrak{m}_\lambda \dotplus \mathfrak{n}_\lambda$, and if λ is $\mathfrak{b}_\mathfrak{k}$-dominant, then $\mathfrak{p}_\lambda \cap \mathfrak{k} = \mathfrak{b}_\mathfrak{k}$. A compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \dotplus \mathfrak{n}$ (i.e., $\mathfrak{p} = \mathfrak{p}_\lambda$ for some $\lambda \in \mathfrak{t}^*$) is *minimal* if it does not properly contain another compatible parabolic subalgebra. It is an important observation that if $\mathfrak{p} = \mathfrak{m} \dotplus \mathfrak{n}$ is minimal, then $\mathfrak{t} \subset Z(\mathfrak{m})$. (Prove!)

A $\mathfrak{k}$-*type* is by definition a simple finite-dimensional $\mathfrak{k}$-module. By $L(\mu)$ we will denote a $\mathfrak{k}$-type with $\mathfrak{b}_\mathfrak{k}$-highest weight μ (μ is then integral for $\mathfrak{k}$ and $\mathfrak{b}_\mathfrak{k}$-dominant). If M is a $(\mathfrak{g}, \mathfrak{k})$-module, we say that a $\mathfrak{k}$-type $L(\mu)$ *is a* $\mathfrak{k}$-*type of* M if $\dim \operatorname{Hom}_\mathfrak{k}(L(\mu), M) \neq 0$. Furthermore, a $\mathfrak{k}$-type $L(\mu)$ of M is called *minimal* if the *Vogan norm*, i.e., the function $\mu' \mapsto \| \operatorname{Re} \mu' + 2\rho\|^2$ defined on the $\mathfrak{b}_\mathfrak{k}$-highest weights μ' of all $\mathfrak{k}$-types of M has a minimum at μ. Any simple $(\mathfrak{g}, \mathfrak{k})$-module M has a minimal $\mathfrak{k}$-type.

Let $\mathfrak{p} = \mathfrak{m} \dotplus \mathfrak{n}$ be a minimal compatible parabolic subalgebra, and let E be a simple finite-dimensional $\mathfrak{p}$-module. Set $\rho_\mathfrak{n} := \rho_{\mathrm{ch}_\mathfrak{t} \mathfrak{n}}$ and $\rho_\mathfrak{n}^\perp := \rho_{\mathrm{ch}_\mathfrak{t}(\mathfrak{n} \cap \mathfrak{k}^\perp)}$. The *fundamental series of* $(\mathfrak{g}, \mathfrak{k})$-*modules* are the modules

$$F^\cdot(\mathfrak{p}, E) := R^\cdot \Gamma_{\mathfrak{k},\mathfrak{t}}(\Gamma_{\mathfrak{t},0}(\operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), E \otimes \Lambda^{\dim \mathfrak{n}}(\mathfrak{n})))).$$

We say that an element $\lambda \in \mathfrak{t}^*$ is $(\mathfrak{g}, \mathfrak{k})$-*regular* if $\langle \operatorname{Re} \lambda, \sigma \rangle \neq 0$ for all $\sigma \in \Delta_\mathfrak{t}$. Let $L(\mu)$ be a $\mathfrak{k}$-type such that $\mu + 2\rho$ is $(\mathfrak{g}, \mathfrak{k})$-regular, and let $\mathfrak{p} = \mathfrak{m} \dotplus \mathfrak{n}$ be the minimal compatible parabolic subalgebra $\mathfrak{p}_{\mu+2\rho}$. We define the $\mathfrak{k}$-type $L(\mu)$ to be *generic* if the following two conditions hold:

(g1) $\langle \operatorname{Re} \mu + 2\rho - \rho_\mathfrak{n}, \alpha \rangle \geq 0\ \forall \alpha \in \operatorname{supp}_\mathfrak{t} \mathfrak{n}_\mathfrak{k}$;
(g2) $\langle \operatorname{Re} \mu + 2\rho - \rho_A, \rho_A \rangle > 0$ for every submultiset A of $\mathrm{ch}_\mathfrak{t} \mathfrak{n}$.

The following is the first main result in this section. It is a shortened version of Theorem 2 in [PZ2].

Theorem 9.1 *Let $L = L(\mu)$ be a fixed generic $\mathfrak{k}$-type, and let $\mathfrak{p} = \mathfrak{m} \dotplus \mathfrak{n}$ be a minimal compatible parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{p}_{\mu+2\rho}$. Let, in addition, E be any simple finite-dimensional $\mathfrak{p}$-module on which $\mathfrak{t}$ acts via the weight $\mu - 2\rho_\mathfrak{n}^\perp$ (E exists since $\mathfrak{t} \subset Z(\mathfrak{m})$). Set $s = \dim \mathfrak{n}_\mathfrak{k}$.*

Then $F^i(\mathfrak{p}, E) = 0$ *for* $i \neq s$, *and the* $(\mathfrak{g}, \mathfrak{k})$*-module* $F^s(\mathfrak{p}, E)$ *is of finite type and is nonzero. Moreover,* L *is the unique minimal* $\mathfrak{k}$*-type of* $F^s(\mathfrak{p}, E)$ *and* $\dim \mathrm{Hom}_{\mathfrak{k}}(L, F^s(\mathfrak{p}, E)) = \dim E$.

Here we outline the proof of a weaker version of Theorem 9.1 in which the vanishing of $F^i(\mathfrak{p}, E)$ is established only for $i > s$. For the stronger result we refer the reader to [PZ2].

This weaker version of Theorem 9.1 is a direct corollary of the following four statements.

Proposition 9.2

(1) Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ *be any minimal compatible parabolic subalgebra, let* E *be any simple finite-dimensional* $\mathfrak{p}$*-module, and let* $L(\delta)$ *be a* $\mathfrak{k}$*-type of* $F^{s-i}(\mathfrak{p}, E)$ *for some* $i \in \mathbb{Z}$. *Then* $i \in \mathbb{Z}_{\geq 0}$ *and there exists* $w \in W_{\mathfrak{k}}$ *of length* i *with respect to the simple roots of* $\mathfrak{b}_{\mathfrak{k}}$, *and a multiset*

$$n_\cdot : \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp) \to \mathbb{Z}_{\geq 0},$$
$$\beta \mapsto n_\beta$$

such that

$$\omega = w(\delta + \rho) - \rho - 2\rho_{\mathfrak{n}}^\perp - \sum_\beta n_\beta \beta,$$

where ω *is the weight via which* $\mathfrak{t}$ *acts on* E.
Furthermore, $\dim \mathrm{Hom}_{\mathfrak{k}}(L(\delta), F^{s-i}(\mathfrak{p}, E))$ *is bounded by the integer*

$$\dim E \left(\sum_{\ell(w)=i} \dim(S^\cdot(\mathfrak{n} \cap \mathfrak{k}^\perp)^{\xi(w)}) \right),$$

where $\xi(w) = w(\delta + \rho) - \rho - \omega - 2\rho_{\mathfrak{n}}^\perp$, *and* $S^\cdot(\mathfrak{n} \cap \mathfrak{k}^\perp)$ *is considered as a* $\mathfrak{k}$*-weight module.*

(2) Under the assumptions of (1)

$$\sum_{0 \leq i \leq s} (-1)^i \dim \mathrm{Hom}_{\mathfrak{k}}(L(\delta), F^{s-i}(\mathfrak{p}, E)) = \tag{9.4}$$
$$\sum_{0 \leq j \leq s} (-1)^j \left(\sum_{m=0}^{\infty} \dim \mathrm{Hom}_{\mathfrak{t}}(H^j(\mathfrak{n} \cap \mathfrak{k}, L(\delta)), \right.$$
$$\left. S^m(\mathfrak{n} \cap \mathfrak{k}^\perp) \otimes E \otimes \Lambda^{\dim(\mathfrak{n} \cap \mathfrak{k}^\perp)}(\mathfrak{n} \cap \mathfrak{k}^\perp)) \right),$$

and the inner sum on the right-hand side of (9.4) is finite.

Proposition 9.2 is a modification of Theorem 6.3.12 and Corollary 6.3.13 in [Vo2], and its proof follows exactly the same lines (an inspection of Vogan's proofs reveals that the symmetry assumption on the pair $(\mathfrak{g}, \mathfrak{k})$ is not needed). Therefore we refer the reader to [Vo2].

Proposition 9.2*(1)* implies that for any minimal compatible parabolic subalgebra $\mathfrak{p}$ and for any simple finite-dimensional $\mathfrak{p}$-module E, the fundamental series module $F^{\cdot}(\mathfrak{p}, E)$ (and thus $F^s(\mathfrak{p}, E)$) is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type, and moreover that $F^i(\mathfrak{p}, E) = 0$ for $i > s$.

In the rest of this section, we assume that $\mathfrak{p}$ and E are as in Theorem 9.1.

Lemma 9.3 *If $L = L(\mu)$ is a $\mathfrak{k}$-type of $F^{s-i}(\mathfrak{p}, E)$, then $i = 0$.*

Proof Choose $\lambda \in \mathfrak{h}^*$ so that $\mathfrak{p} = \mathfrak{p}_\lambda$. Then $\langle \mathrm{Re}\,\lambda, \gamma \rangle > 0$ for $\gamma \in \mathrm{supp}_{\mathfrak{t}}\,\mathfrak{n}$. By Proposition 9.2*(1)*, there exist $w \in W_{\mathfrak{k}}$ of length i and a multiset

$$n_\cdot : \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp) \to \mathbb{Z}_{\geq 0}$$

such that

$$\omega = w(\mu + \rho) - \rho - 2\rho_{\mathfrak{n}}^{\perp} - \sum_{\beta \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp)} n_\beta \beta.$$

In addition, $\omega = \mu - 2\rho_{\mathfrak{n}}^{\perp}$ by hypothesis. Hence

$$w(\mu + \rho) - (\mu + \rho) = \sum_{\beta \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp)} n_\beta \beta.$$

On the other hand since $\mu + \rho$ is $\mathfrak{b}_{\mathfrak{k}}$-dominant, there exists a multiset

$$m_\cdot : \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}) \to \mathbb{Z}_{\geq 0}$$

such that $(\mu + \rho) - w(\mu + \rho) = \sum_{\alpha \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k})} m_\alpha \alpha$. Therefore

$$\sum_{\alpha \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k})} m_\alpha \alpha + \sum_{\beta \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp)} n_\beta \beta = 0$$

and

$$\sum_{\alpha \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k})} m_\alpha \langle \mathrm{Re}\,\lambda, \alpha \rangle + \sum_{\beta \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp)} n_\beta \langle \mathrm{Re}\,\lambda, \beta \rangle = 0.$$

Hence $m_\alpha = n_\beta = 0$ for all $\alpha, \beta \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^\perp)$, and so $w(\mu + \rho) = \mu + \rho$. As $\mu + \rho$ is a regular weight of $\mathfrak{k}$, we get $w = \mathrm{id}$ and hence $i = 0$. □

The following two lemmas prove the last claim of Theorem 9.1.

Lemma 9.4 $\dim \mathrm{Hom}_{\mathfrak{k}}(L, F^s(\mathfrak{p}, E)) = \dim E$.

Proof Lemma 9.3 enables us to rewrite (9.4) in the special case $\delta = \mu$ as

$$\dim \mathrm{Hom}_{\mathfrak{k}}(L(\mu), F^s(\mathfrak{p}, E)) = \sum_{0\le j\le s} (-1)^j \left(\sum_{m=0}^{\infty} \dim \mathrm{Hom}_{\mathfrak{t}}(H^j(\mathfrak{n}\cap\mathfrak{k}, L(\mu)), S^m(\mathfrak{n}\cap\mathfrak{k}^\perp)\otimes E\otimes \Lambda^{\dim(\mathfrak{n}\cap\mathfrak{k}^\perp)}(\mathfrak{n}\cap\mathfrak{k}^\perp))\right),$$

and by a well-known theorem of Kostant [Ko1],

$$\mathrm{supp}_{\mathfrak{t}}\, H^{\cdot}(\mathfrak{n}\cap\mathfrak{k}, L(\mu)) = \{\tilde{\sigma}(\mu+\rho)-\rho \mid \tilde{\sigma}\in W_{\mathfrak{k}}\}$$

and μ appears with multiplicity 1 in $\{\tilde{\sigma}(\mu+\rho)-\rho \mid \tilde{\sigma}\in W_{\mathfrak{k}}\}$. On the other hand,

$$\mathrm{supp}_{\mathfrak{t}}(S^{\cdot}(\mathfrak{n}\cap\mathfrak{k}^\perp)\otimes E\otimes\Lambda^{\dim(\mathfrak{n}\cap\mathfrak{k}^\perp)}(\mathfrak{n}\cap\mathfrak{k}^\perp)) = \left\{\mu + \sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta\,\beta \mid n_\beta\in\mathbb{Z}_{\ge 0}\right\}.$$

Since $\mu+\rho$ is $\mathfrak{b}_{\mathfrak{k}}$-dominant,

$$\{\tilde{\sigma}(\mu+\rho)-\rho \mid \tilde{\sigma}\in W_{\mathfrak{k}}\} \subset \left\{\mu - \sum_{\alpha\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k})} m_\alpha\alpha \mid m_\alpha\in\mathbb{Z}_{\ge 0}\right\}.$$

This, together with the inequality $\langle \mathrm{Re}\,\lambda, \gamma\rangle > 0$ for all $\gamma\in\mathrm{supp}_{\mathfrak{t}}\,\mathfrak{n}$ (see the proof of Lemma 9.3), allows us to conclude that

$$\{\tilde{\sigma}(\mu+\rho)-\rho \mid \tilde{\sigma}\in W_{\mathfrak{k}}\} \cap \left\{\mu + \sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta\beta\right\} = \{\mu\}.$$

Consequently,

$$\mathrm{Hom}_{\mathfrak{t}}(H^j(\mathfrak{n}\cap\mathfrak{k}, L(\mu)), S^m(\mathfrak{n}\cap\mathfrak{k}^\perp)\otimes E\otimes\Lambda^{\dim(\mathfrak{n}\cap\mathfrak{k}^\perp)}(\mathfrak{n}\cap\mathfrak{k}^\perp)) \neq 0 \qquad (9.5)$$

only for $m = 0$. Since $\mathfrak{k}$ acts on $E \otimes \Lambda^{\dim(\mathfrak{n}\cap\mathfrak{k}^\perp)}$ via the weight μ, Kostant's Theorem and (9.5) imply also $j = 0$. This yields

$$\begin{aligned}\dim \mathrm{Hom}_{\mathfrak{k}}(L(\mu), F^s(\mathfrak{p}, E)) &= \dim \mathrm{Hom}_{\mathfrak{t}}(H^0(\mathfrak{n}\cap\mathfrak{k}, L(\mu)), E \otimes \Lambda^{\dim(\mathfrak{n}\cap\mathfrak{k}^\perp)}(\mathfrak{n}\cap\mathfrak{k}^\perp)) \\ &= \dim E.\end{aligned}$$

□

Lemma 9.5 *If $L(\delta)$ is a $\mathfrak{k}$-type of $F^s(\mathfrak{p}, E)$ and $\delta \neq \mu$, then*

$$\|\mathrm{Re}\,\delta + 2\rho\| > \|\mathrm{Re}\,\mu + 2\rho\|.$$

Proof By Proposition 9.2, there exists a multiset $n_\cdot : \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp) \to \mathbb{Z}_{\geq 0}$ such that

$$\delta + \rho = \mu + \rho + \sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta \beta.$$

Hence

$$\delta + 2\rho = \mu + 2\rho + \sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta \beta.$$

Since $\mathfrak{p} = \mathfrak{p}_{\mu+2\rho}$, the inequality $\langle \mathrm{Re}\,\mu + 2\rho, \beta\rangle \geq 0$ holds for all $\beta \in \mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)$. In addition, $\delta \neq \mu$ implies $\| \sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta \beta\|^2 > 0$. Therefore

$$\begin{aligned}\|\mathrm{Re}\,\delta + 2\rho\|^2 = \|\mathrm{Re}\,\mu + 2\rho\|^2 + \| &\sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta \beta\|^2 \\ &+2 \sum_{\beta\in\mathrm{supp}_{\mathfrak{t}}(\mathfrak{n}\cap\mathfrak{k}^\perp)} n_\beta \langle \mathrm{Re}\,\mu + 2\rho, \beta\rangle > \|\mathrm{Re}\,\mu + 2\rho\|^2.\end{aligned}$$

□

Theorem 9.1 leads to the following.

Corollary 9.6 *For any pair $(\mathfrak{g}, \mathfrak{k})$ and any generic $\mathfrak{k}$-type L, there exists a simple $(\mathfrak{g}, \mathfrak{k})$-module of finite type with unique minimal $\mathfrak{k}$-type L.*

Proof The module whose existence is claimed by Corollary 9.6 can be constructed as any simple quotient of a $\mathfrak{g}$-submodule of $F^s(\mathfrak{p}, E)$ generated by the image of any $\mathfrak{k}$-module injection $L \to F^s(\mathfrak{p}, E)$. □

Corollary 9.6 doesn't quite answer our starting question for this section as it does not state the existence of an infinite-dimensional simple $(\mathfrak{g}, \mathfrak{k})$-module of finite type. The answer is given by the following theorem.

Theorem 9.7 *For an algebraic reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$, there exists a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module of finite type if and only if $\mathfrak{k}$ is not an ideal of $\mathfrak{g}$.*

We briefly explain the idea of the proof and refer the reader to [PZ6] for the details. First, it is not difficult to prove that if $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, then any simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module has infinite type over $\mathfrak{k}$. Therefore in this case the module from Corollary 9.6 must be finite dimensional.

If $\mathfrak{k}$ is not an ideal of $\mathfrak{g}$, it turns out that the simple $(\mathfrak{g}, \mathfrak{k})$-module $F^s(\mathfrak{p}, E)$ is infinite dimensional for an appropriate choice of μ and E. Indeed, if the central character of $F^s(\mathfrak{p}, E)$ can be chosen non-integral, we are done. In the case when the conditions of Theorem 9.1 force the central character of $F^s(\mathfrak{p}, E)$ to be integral, it is shown in [PZ6] that the $\mathfrak{p}$-highest weight of E can be chosen dominant and integral for $\mathfrak{g}$, and that in this case $\dim F^s(\mathfrak{p}, E) = \infty$.

Finally, we would like to point out that Corollary 9.6 is valid without the genericity assumption on L. This has been proved in [PZ3].

9.2.2 Reconstruction Theorems

A natural next question is: do all simple $(\mathfrak{g}, \mathfrak{k})$-module s arise through the construction of Theorem 9.1? It is well known from Harish-Chandra module theory that the answer is "no". Moreover, a general simple $(\mathfrak{g}, \mathfrak{k})$-module of finite type may have several minimal $\mathfrak{k}$-types. This applies for instance to the Britten–Lemire modules (see Theorem 8.19). To the reader: please identify a simple weight $\mathfrak{sl}(2)$-module with two minimal $\mathfrak{h}$-types!

For a symmetric subalgebra $\mathfrak{k}$, there is a celebrated classification of all simple $(\mathfrak{g}, \mathfrak{k})$-modules (or Harish-Chandra modules) under the assumption that $\mathfrak{k}$ is the Lie algebra of a symmetric subgroup K of a reductive algebraic group G with Lie algebra $\mathfrak{g}$, see [Vo2, BB]. As we already mentioned in Sect. 8.2.6 for $\mathfrak{k} = \mathfrak{h}$, all simple $(\mathfrak{g}, \mathfrak{h})$-modules of finite type, i.e., all simple weight modules with finite-dimensional weight spaces, have been classified (see [Mat] and the references therein).

There is no general classification of $(\mathfrak{g}, \mathfrak{k})$-module s of finite type beyond the cases when $\mathfrak{k} = \mathfrak{h}$ or $\mathfrak{k}$ is symmetric, and in general it is not known what are all $(\mathfrak{g}, \mathfrak{k})$-module s with a fixed minimal $\mathfrak{k}$-type L. However, if one assumes that L is generic, one can show that any simple $(\mathfrak{g}, \mathfrak{k})$-module arises in the fundamental series through the construction of Corollary 9.6. This result is proved in [PZ2] and is inspired by a result of Vogan about Harish-Chandra modules [Vo1].

Theorem 9.8 below provides an explicit procedure that finds the "fundamental series data" $(\mathfrak{p}, E)$ corresponding to any simple generalized Harish-Chandra module with a minimal $\mathfrak{k}$-type isomorphic to L. This procedure resembles the process of

finding the highest weight of a simple module about which it is known that it has a highest weight but this weight is not given. In this latter case, the highest weight space of M is the space of $\mathfrak{n}$-coinvariants of M, i.e., $H^0(\mathfrak{n}, M)$, where $\mathfrak{n}$ is the nilpotent radical of the respective Borel subalgebra. In the case of a generic simple $(\mathfrak{g}, \mathfrak{k})$-module M, the $\mathfrak{p}$-module E turns out to be a weight space in a higher cohomology group of the nilradical $\mathfrak{n}$ of $\mathfrak{p}$ with coefficients in M. In this sense, simple $(\mathfrak{g}, \mathfrak{k})$-module s with generic minimal $\mathfrak{k}$-type strongly resemble highest weight modules (without being such).

Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be a compatible parabolic subalgebra. Then we note that any cohomology space $H^i(\mathfrak{n}, M)$ for a $(\mathfrak{g}, \mathfrak{k})$-module M is an $\mathfrak{m}$-module and a $\mathfrak{t}$-weight module. In what follows, we write $H^r(\mathfrak{n}, M)^\eta$ for the respective weight space of $\eta \in \mathfrak{t}^*$.

We are now able to state two reconstruction theorems proved in [PZ2].

Theorem 9.8 (First Reconstruction Theorem) *Let M be a simple $(\mathfrak{g}, \mathfrak{k})$-module of finite type with a minimal $\mathfrak{k}$-type $L(\mu)$ that is generic. Then $\mathfrak{p} := \mathfrak{p}_{\mu+2\rho} = \mathfrak{m} \oplus \mathfrak{n}$ is a minimal compatible parabolic subalgebra. Let $\omega := \mu - 2\rho_{\mathfrak{n}}^{\perp}$ (recall that $\rho_{\mathfrak{n}}^{\perp} = \rho_{\mathrm{ch}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^{\perp})}$) and let E be the $\mathfrak{p}$-module $H^r(\mathfrak{n}, M)^\omega$ with trivial $\mathfrak{n}$-action, where $r = \dim(\mathfrak{n} \cap \mathfrak{k}^{\perp})$.*

Then E is a simple $\mathfrak{p}$-module and M is isomorphic to the $(\mathfrak{g}, \mathfrak{k})$-submodule $\bar{F}^s(\mathfrak{p}, E)$ of $F^s(\mathfrak{p}, E)$ for $s = \dim(\mathfrak{n} \cap \mathfrak{k})$, generated by the isotypic component of the unique minimal $\mathfrak{k}$-type of $F^s(\mathfrak{p}, E)$.

Theorem 9.9 (Second Reconstruction Theorem) *Assume that $\mathfrak{k} \subset \mathfrak{g}$ is a reductive in $\mathfrak{g}$ subalgebra that contains a regular element of $\mathfrak{g}$. Let M be a simple $(\mathfrak{g}, \mathfrak{k})$-module (a priori of infinite type) with a minimal $\mathfrak{k}$-type $L(\mu)$ that is generic. Then M has finite $\mathfrak{k}$-type, and hence Theorem 9.8 applies to M.*

Theorem 9.8 is based on the following.

Theorem 9.10 *Let M be a finite-length $(\mathfrak{g}, \mathfrak{k})$-module that has a generic minimal $\mathfrak{k}$-type $L(\mu)$. There exists a vector space isomorphism*

$$(M^{\mathfrak{n} \cap \mathfrak{k}})^\mu \otimes \Lambda^r(\mathfrak{n} \cap \mathfrak{k}^{\perp})^* \cong H^r(\mathfrak{n}, M)^{\mu - 2\rho_{\mathfrak{n}}^{\perp}}, \tag{9.6}$$

where $\mathfrak{n} := \mathfrak{n}_{\mu+2\rho}$. Moreover,

$$H^i(\mathfrak{n}, M)^{\mu - 2\rho_{\mathfrak{n}}^{\perp}} = 0$$

for $i \neq r$.

In order to outline the proof of Theorem 9.10, we first present some results on $\mathfrak{n}$-cohomology. For an introduction to Lie algebra cohomology, see [Wei].

Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be a compatible parabolic subalgebra corresponding to an element $\lambda \in \mathfrak{t}^*$ (i.e., $\mathfrak{p} = \mathfrak{p}_\lambda$) which we assume $\mathfrak{k}$-regular, and let M be a finite-length $(\mathfrak{g}, \mathfrak{k})$-module.

Proposition 9.11 *In the category of* $\mathfrak{t}$*-weight modules, there exists a bounded (not necessarily first quadrant) cohomology spectral sequence that converges to* $H^{\cdot}(\mathfrak{n}, M)$*, with*

$$E_1^{a,b} = H^{a+b-R(a)}(\mathfrak{n} \cap \mathfrak{k}, M) \otimes V_a^*,$$

where a *runs over* $\{0, \ldots, n\}$ *for some* n*,* R *is a monotonic function on* $\{0, \ldots, n\}$ *with values in* $\mathbb{Z}_{\geq 0}$ *such that* $R(a) \leq a$ *and* $R(n) = r$*,* V_a *is a* $\mathfrak{t}$*-submodule of* $\Lambda^{R(a)}(\mathfrak{n} \cap \mathfrak{k}^{\perp})$ *for every* a*, and* $V_n = \Lambda^r(\mathfrak{n} \cap \mathfrak{k}^{\perp})$*.*

The spectral sequence whose existence is claimed in Proposition 9.11 is a version of the Hochschild–Serre spectral sequence and is constructed explicitly by Vogan in [Vo2, Theorem 5.2.2] under the assumption that the pair $(\mathfrak{g}, \mathfrak{k})$ is symmetric. However, as this assumption is not used in the construction, we refer the reader directly to [Vo2].

Proposition 9.11 has the following corollary.

Corollary 9.12 *For each* $\kappa \in \mathfrak{t}^*$*, we have a spectral sequence of vector spaces that converges to* $H^{\cdot}(\mathfrak{n}, M)^{\kappa}$ *and whose* E_1*-term is* $(E_1^{a,b})^{\kappa}$*, where* $E_1^{a,b}$ *is as in Proposition 9.11. Moreover, there are (edge) homomorphisms*

$$\pi_i^{\kappa+2\rho_{\mathfrak{n}}^{\perp}} : H^i(\mathfrak{n} \cap \mathfrak{k}, M)^{\kappa+2\rho_{\mathfrak{n}}^{\perp}} \otimes \Lambda^r(\mathfrak{n} \cap \mathfrak{k}^{\perp})^* \to H^{i+r}(\mathfrak{n}, M)^{\kappa}, \tag{9.7}$$

where $i = n + b - r$*.*

Proof The fact that the spectral sequence of Proposition 9.11 is a spectral sequence in the category of weight $\mathfrak{t}$-modules implies that it has a well-defined direct summand consisting of κ-weight vectors. Its corresponding $E_1^{a,b}$-term equals $(E_1^{a,b})^{\kappa}$.

In [Vo2, 5.2], Vogan constructs (under the assumption that the pair $(\mathfrak{g}, \mathfrak{k})$ is symmetric) linear maps $(E_1^{n,b})^{\kappa} \to (E_{\infty}^{n,b})^{\kappa}$ which in turn yield edge homomorphisms for the spectral sequence with term $(E_1^{a,b})^{\kappa}$,

$$\pi_i^{\kappa+2\rho_{\mathfrak{n}}^{\perp}} : H^i(\mathfrak{n} \cap \mathfrak{k}, M)^{\kappa+2\rho_{\mathfrak{n}}^{\perp}} \otimes \Lambda^r(\mathfrak{n} \cap \mathfrak{k}^{\perp})^* \to H^{i+r}(\mathfrak{n}, M)^{\kappa} \cong \bigoplus_{a+b=i+r} (E_{\infty}^{a,b})^{\kappa}.$$

This construction extends to the more general case we consider, and we refer the reader to [Vo2]. □

Proposition 9.13 *Under the hypothesis of Theorem 9.10,* $(E_1^{a,b})^{\mu-2\rho_{\mathfrak{n}}^{\perp}} = 0$ *for* $(a, b) \neq (n, r - n)$*; therefore the spectral sequence from Corollary 9.12 for* $\kappa = \mu - 2\rho_{\mathfrak{n}}^{\perp}$ *collapses at the level* E_1*.*

Proof Let $L(\delta)$ be a $\mathfrak{k}$-type such that

$$(H^i(\mathfrak{n} \cap \mathfrak{k}, L(\delta)) \otimes \Lambda^j(\mathfrak{n} \cap \mathfrak{k}^{\perp})^*)^{\mu-2\rho_{\mathfrak{n}}^{\perp}} \neq 0$$

for some $(i, j) \neq (0, r)$. Then according to [Vo2, 5.4], there exist $\sigma \in W_{\mathfrak{k}}$ and a submultiset A of $\mathrm{ch}_{\mathfrak{t}}(\mathfrak{n} \cap \mathfrak{k}^{\perp})$ such that $\sigma(\delta+\rho)-\rho = \mu - 2\rho_A$. Hence $\sigma(\delta+2\rho) = \mu+\rho-2\rho_A+\sigma\rho = \mu+2\rho-(2\rho_A+\rho-\sigma\rho) = \mu+2\rho-2\rho_B$ for an appropriate submultiset B of $\mathrm{ch}_{\mathfrak{t}}\,\mathfrak{n}$. Furthermore, as $L(\mu)$ is generic, $\langle \mathrm{Re}\,\mu+2\rho-\rho_B, \rho_B\rangle > 0$ by condition (g2), and thus

$$\begin{aligned}\|\mathrm{Re}\,\delta+2\rho\|^2 &= \|\sigma(\mathrm{Re}\,\delta+2\rho)\|^2 \\ &= \|\mathrm{Re}\,\mu+2\rho\|^2 - 4\langle \mathrm{Re}\,\mu+2\rho-\rho_B, \rho_B\rangle < \|\mathrm{Re}\,\mu+2\rho\|^2.\end{aligned}$$

Since $L(\mu)$ is minimal in M, $L(\delta)$ is not a $\mathfrak{k}$-type of M, and hence

$$(H^i(\mathfrak{n}\cap\mathfrak{k}, M)\otimes\Lambda^j(\mathfrak{n}\cap\mathfrak{k}^{\perp})^*)^{\mu-2\rho_{\mathfrak{n}}^{\perp}} = 0$$

for $(i, j) \neq (0, r)$. Since $(E_1^{a,b})^{\mu-2\rho_{\mathfrak{n}}^{\perp}} \subset (H^i(\mathfrak{n}\cap\mathfrak{k}, M)\otimes\Lambda^j(\mathfrak{n}\cap\mathfrak{k}^{\perp})^*)^{\mu-2\rho_{\mathfrak{n}}^{\perp}}$ for $i = a+b-R(a)$ and $j = R(a)$, we obtain

$$(E_1^{a,b})^{\mu-2\rho_{\mathfrak{n}}^{\perp}} = 0$$

for $(a, b) \neq (n, r-n)$. □

Proof of Theorem 9.10 Theorem 9.10 follows from Proposition 9.13 via two observations: first,

$$H^i(\mathfrak{n}, M)^{\mu-2\rho_{\mathfrak{n}}^{\perp}} \cong \bigoplus_{a+b=i}(E_\infty^{a,b})^{\mu-2\rho_{\mathfrak{n}}^{\perp}} = \bigoplus_{a+b=i}(E_1^{a,b})^{\mu-2\rho_{\mathfrak{n}}^{\perp}}$$

and hence $H^i(\mathfrak{n}, M)^{\mu-2\rho_{\mathfrak{n}}^{\perp}} = 0$ for $i \neq r$, and second, the desired isomorphism (9.6) is nothing but the edge isomorphism π_0^{μ}. □

We are now ready to prove Theorem 9.8.

Proof of Theorem 9.8 We first need to refer to Corollary 3 in [PZ2] to claim that $H^{\cdot}(\mathfrak{n}, M)$ is a finite-dimensional $\mathfrak{t}$-weight $\mathfrak{m}$-module. By Theorem 9.10, $H^r(\mathfrak{n}, M)^{\omega} \neq 0$. Let $\tilde{E}$ be any simple quotient of the $\mathfrak{m}$-module $H^r(\mathfrak{n}, M)^{\omega}$. Consider $\tilde{E}$ as a simple $\mathfrak{p}$-module by letting $\mathfrak{n}$ act trivially on $\tilde{E}$.

The fact that μ is generic implies that the pair $(\mathfrak{p}, \tilde{E})$ satisfies the hypotheses of Theorem 9.8. Thus $F^s(\mathfrak{p}, \tilde{E}) \neq 0$ and there is a canonical isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(M, F^s(\mathfrak{p}, \tilde{E})) \cong \mathrm{Hom}_{\mathfrak{m}}(H^r(\mathfrak{n}, M), \tilde{E}). \tag{9.8}$$

Hence the surjection $H^r(\mathfrak{n}, M) \to \tilde{E}$ determines via (9.8) a canonical $\mathfrak{g}$-module isomorphism

$$M \cong \bar{F}^s(\mathfrak{p}, \tilde{E}). \tag{9.9}$$

Therefore by Theorem 9.1 $\tilde{E}$ is isomorphic to $H^r(\mathfrak{n}, M)^\omega$, and the surjection $H^r(\mathfrak{n}, M)^\omega \to \tilde{E}$ can be chosen as the identity map. This implies finally that the isomorphism (9.9) is a canonical isomorphism

$$M \cong \bar{F}^s(\mathfrak{p}, E)$$

for $E = H^r(\mathfrak{n}, M)^\omega$, as required. □

As a simple illustration of the first and second reconstruction theorems, we briefly discuss the case when $\mathfrak{k}$ is an $\mathfrak{sl}(2)$-subalgebra.

If $\mathfrak{k} \simeq \mathfrak{sl}(2)$, then $\dim \mathfrak{t} = 1$, and for any $\alpha \in \Delta_\mathfrak{t}$, $\alpha = a\rho$, where $a := \frac{\langle \alpha, \rho \rangle}{\|\rho\|^2} > 0$. Moreover $\mu = \omega + 2\rho_\mathfrak{n}^\perp = m\rho$ for $m \in \mathbb{Z}_{\geq 0}$, and the genericity condition is equivalent to the single inequality

$$\langle \mu + 2\rho - \rho_\mathfrak{n}, \rho \rangle > 0,$$

or to the inequality

$$\dim L(\mu) = m + 1 \geq \tilde{\rho}(h),$$

where h is the semisimple element of the canonical $\mathfrak{sl}(2)$-basis e, f, h in $\mathfrak{k}$ with $h \in \mathfrak{h}$, and $\tilde{\rho} := \rho_{\mathrm{ch}_\mathfrak{h}\, \mathfrak{b}}$. The integer $\tilde{\rho}(h)$ depends on the pair $(\mathfrak{g}, \mathfrak{k})$ and can be computed in the following way.

Write $\tilde{\rho} = \sum_i r_i \alpha_i$, where $\alpha_i \in \mathfrak{h}^*$ are the simple roots of $\mathfrak{b}$. The non-negative half-integers r_i are well known, see [B]. Furthermore, a result of E. Dynkin [Dyn], states that $\alpha_i(h) \in \{0, 1, 2\}$, and that $\mathfrak{k}$ is a principal $\mathfrak{sl}(2)$-subalgebra if $\alpha_i(h) = 2$ for all i. The final inequality which is equivalent to the genericity of $L(\mu)$ becomes

$$\dim L(\mu) = m + 1 \geq \sum_i \alpha_i(h) r_i. \tag{9.10}$$

In particular, for a principal $\mathfrak{sl}(2)$-subalgebra it reads

$$m + 1 \geq 2\left(\sum_i r_i\right). \tag{9.11}$$

Theorem 9.9 implies, in particular, that any simple $(\mathfrak{g}, \mathfrak{k})$-module with minimal $\mathfrak{k}$-type $L(\mu)$ satisfying (9.10) automatically has finite type, or in other words, that if $L(\mu)$ is the minimal $\mathfrak{k}$-type of a simple $(\mathfrak{g}, \mathfrak{k})$-module of infinite type, then $\dim L(\mu)$ does not exceed the right-hand side of (9.10). Note however, that (9.10) is only a sufficient but not a necessary condition for every simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module with minimal $\mathfrak{k}$-type $L(\mu)$ to necessarily be of finite type. ◯

If $\mathfrak{g} = \mathfrak{sl}(3)$ and $\mathfrak{k}$ is a principal $\mathfrak{sl}(2)$-subalgebra, then the pair $(\mathfrak{g}, \mathfrak{k})$ is simply the symmetric pair $(\mathfrak{sl}(3), \mathfrak{o}(3))$, and so any simple $(\mathfrak{g}, \mathfrak{k})$-module has finite type. Moreover, inequality (9.11) which in this case is nothing but

$$m \geq 3,$$

is the necessary and sufficient condition for the first reconstruction theorem to hold.

If $\mathfrak{g} = \mathfrak{o}(5)$ and $\mathfrak{k}$ is a principal $\mathfrak{sl}(2)$-subalgebra, (9.11) is equivalent to

$$m \geq 6.$$

Nevertheless, one can show (unpublished work of the first author with Zuckerman) that for $m = 5$ the claims of both Theorem 9.8 and Theorem 9.9 still hold. In this way, Theorem 9.8 describes all simple $(\mathfrak{g}, \mathfrak{k})$-modules with minimal $\mathfrak{k}$-types $L(r\rho)$ for $r \geq 5$. In particular, the multiplicity of the minimal $\mathfrak{k}$-type of such a module necessarily equals 1. The case when $m \leq 4$ is the "smallest" case when the problem of classifying all simple $(\mathfrak{g}, \mathfrak{k})$- modules of finite type is still open.

In the paper [PS7], all $(\mathfrak{o}(5), \mathfrak{sl}(2))$-modules with bounded $\mathfrak{sl}(2)$-multiplicities are classified. It turns out that the minimal $\mathfrak{sl}(2)$-type of such a module is $L(0)$, $L(\rho)$, $L(2\rho)$, $L(3\rho)$ or $L(4\rho)$ and that its multiplicity can be higher than one (except for $L(4\rho)$). Hence no such module is generic, or equivalently, every generic $(\mathfrak{o}(5), \mathfrak{sl}(2))$-modules has unbounded $\mathfrak{sl}(2)$-multiplicities.

9.2.3 *Fernando–Kac Subalgebra of a Fundamental Series Module*

Next, we would like to address briefly one more natural direction of study of generalized Harish-Chandra modules in the finite-dimensional case. This is generally speaking the relation between $\mathfrak{k}$ and $\mathfrak{g}[M]$ for a simple $(\mathfrak{g}, \mathfrak{k})$-module. Clearly, $\mathfrak{k} \subset \mathfrak{g}[M]$, but how large is $\mathfrak{g}[M]$? A closely related question is the following modification of our starting question in this chapter: given a semisimple Lie algebra $\mathfrak{g}$ and a reductive in $\mathfrak{g}$ Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$, does there necessarily exist a simple $(\mathfrak{g}, \mathfrak{s})$-module M of finite type such that $\mathfrak{s}$ is the reductive part of $\mathfrak{g}[M]$, i.e., $\mathfrak{g}[M] = \mathfrak{n} \dotplus \mathfrak{s}$, where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{g}[M]$? The answer to this question is "no". In fact, in [PSZ1] a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is defined to be *primal* if a $(\mathfrak{g}, \mathfrak{s})$-module M as above does exist. More generally, a Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a *Fernando–Kac subalgebra of finite type* if $\mathfrak{g}$ admits an irreducible $\mathfrak{g}$-module M with $\mathfrak{g}[M] = \mathfrak{l}$ that is of finite type over $\mathfrak{l}$.

In what follows, we call a $(\mathfrak{g}, \mathfrak{k})$-module *strict* if $\mathfrak{k} = \mathfrak{g}[M]$.

Some main results of [PSZ1] are summarized in the following.

Theorem 9.14 *Let $\mathfrak{g}$ be a (finite-dimensional) reductive Lie algebra.*

(1) If a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a Fernando–Kac subalgebra of finite type, then $\mathfrak{l}$ is self-normalizing (and hence $\mathfrak{l}$-algebraic) and admits a decomposition $\mathfrak{n}_\mathfrak{l} \dotplus \mathfrak{l}_{red}$, where $\mathfrak{n}_\mathfrak{l}$ is the nilradical of $\mathfrak{l}$ and $\mathfrak{l}_{red}$ is a reductive in $\mathfrak{g}$ subalgebra.

(2) A reductive in $\mathfrak{g}$ subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is primal if and only if it is self-normalizing. This is equivalent to the condition $C_\mathfrak{g}(\mathfrak{s}) = Z_\mathfrak{s}$, where $C_\mathfrak{g}(\mathfrak{s})$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$ and $Z_\mathfrak{s}$ is the center of $\mathfrak{s}$.

Comment on the Proof We will prove *(1)* and the easier direction of *(2)*.

(1) Let M be a strict simple $(\mathfrak{g}, \mathfrak{l})$-module and let $M_0 \subset M$ be a simple (finite-dimensional) $\mathfrak{l}$-submodule. Assume, to the contrary, that $N_\mathfrak{g}(\mathfrak{l}) \neq \mathfrak{l}$. Then one can choose $x \in N_\mathfrak{g}(\mathfrak{l}) \backslash \mathfrak{l}$ such that $[x, \mathfrak{l}_{ss}] = 0$ for a fixed Levi decomposition $\mathfrak{l} = \mathfrak{l}_{ss} \dotplus \mathfrak{r}_\mathfrak{l}$. Since $x \notin \mathfrak{l}$, x acts freely on M. Set

$$M_n := M_0 + x \cdot M_0 + \cdots + x^n \cdot M_0.$$

A simple calculation using $[x, \mathfrak{l}_{ss}] = 0$ and $[x, \mathfrak{r}_\mathfrak{l}] \subset \mathfrak{r}_\mathfrak{l}$ shows that M_n is $\mathfrak{l}$-stable and M_n/M_{n-1} is isomorphic to M_0 as a $\mathfrak{l}$-module. Therefore the multiplicity of M_0 in M is infinite. Since this is a contradiction, we have $N_\mathfrak{g}(\mathfrak{l}) = \mathfrak{l}$.

By [B, Ch.VII, §5, Cor.1], any self-normalizing subalgebra is splittable, and by [B, Ch.VII, §5, Prop.7] any splittable subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ admits a decomposition $\mathfrak{l} = \mathfrak{l}_{red} \dotplus \mathfrak{n}_\mathfrak{l}$ as required. This proves *(1)*.

(2) If $\mathfrak{l} = \mathfrak{l}_{red} \dotplus \mathfrak{n}_\mathfrak{l}$ and $N_\mathfrak{g}(\mathfrak{l}) = \mathfrak{l}$, it is easy to see that $N_\mathfrak{g}(\mathfrak{l}_{red}) = \mathfrak{l}_{red}$. Therefore a primal in $\mathfrak{g}$ subalgebra $\mathfrak{s} = \mathfrak{l}_{red}$ is self-normalizing, and (as $\mathfrak{s}$ is reductive in $\mathfrak{g}$), $N_\mathfrak{g}(\mathfrak{s}) = C_\mathfrak{g}(\mathfrak{s}) + \mathfrak{s} = \mathfrak{s}$, hence $C_\mathfrak{g}(\mathfrak{s}) = Z_\mathfrak{s}$.

To finish the proof of *(2)*, one needs to show that for any $\mathfrak{s}$ as above, there is a simple $(\mathfrak{g}, \mathfrak{l})$-module M of finite type such that $\mathfrak{g}[M] = \mathfrak{s} \dotplus \mathfrak{n}_{\mathfrak{g}[M]}$. The construction of M uses $\mathscr{D}$-modules and we refer the reader to the original article [PSZ1].

□

Let's now briefly discuss the Fernando–Kac subalgebra of an infinite-dimensional simple $(\mathfrak{g}, \mathfrak{k})$-module M with generic minimal $\mathfrak{k}$-type $L(\mu)$. If $\mathfrak{k}$ is a maximal proper subalgebra of $\mathfrak{g}$, then clearly $\mathfrak{g}[M] = \mathfrak{k}$ as $\mathfrak{g}[M]$ contains $\mathfrak{k}$ by definition, and $\mathfrak{g}[M] \neq \mathfrak{g}$ by assumption. For instance, a principal $\mathfrak{sl}(2)$-subalgebra is maximal in $\mathfrak{o}(k)$ for $k \geq 5$, $k \neq 7$, and in $\mathfrak{sp}(2k)$ for $k \geq 2$, see [Dyn, Theorem 1.5]. The principal $\mathfrak{sl}(2)$-subalgebra is never maximal in $\mathfrak{sl}(n)$ for $n \geq 3$, however the only intermediate subalgebra is either $\mathfrak{o}(n)$ or $\mathfrak{sp}(2n)$, depending on whether n is even or odd. (To the reader: prove this claim!) Furthermore, J. Willenbring and G. Zuckerman have proved the following remarkable result.

Theorem 9.15 ([WZ]) *Let $\mathfrak{g} = \mathfrak{o}(n), \mathfrak{sp}(2n)$ and let $\mathfrak{k} \simeq \mathfrak{sl}(2)$ be a principal $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$. Then there are finitely many $\mathfrak{k}$-types such that any $\mathfrak{g}$-type must contain one of these $\mathfrak{k}$-types.*

Theorem 9.15 implies that any simple $(\mathfrak{sl}(n), \mathfrak{o}(n))$-module or $(\mathfrak{sl}(n), \mathfrak{sp}(2n))$-module of finite type is a $(\mathfrak{sl}(n), \mathfrak{sl}(2))$-module of infinite type, $\mathfrak{sl}(2)$ being a principal $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{o}(n)$ or $\mathfrak{sp}(2n)$ (i.e., in particular a principal $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{sl}(n)$). Prove this claim! Therefore we have the following.

Corollary 9.16 *If $\mathfrak{k} \simeq \mathfrak{sl}(2)$ is a principal $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{o}(n)$ (here $n \neq 7$), $\mathfrak{sp}(2n)$ and M is a simple $(\mathfrak{g}, \mathfrak{k})$-module of finite type, then $\mathfrak{g}[M] = \mathfrak{k}$.*

For a general reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$, Zuckerman and the first author have the following conjecture.

Conjecture 9.17 Let $M = \bar{F}^s(\mathfrak{p}, E)$ be as in Theorem 9.8. If M is a simple $\mathfrak{g}$-module, then

$$\mathfrak{g}[M] = \mathfrak{k}_{ss} \ni \mathfrak{p}_{red,\mathfrak{k}},$$

where $\mathfrak{p}_{red,\mathfrak{k}}$ is the maximal $\mathfrak{k}$-stable subspace of $\mathfrak{p}_{red}$.

The claim of this conjecture could follow from an appropriate comparison theorem between the Zuckerman functor and the $\mathscr{D}$-module construction of [PSZ1] (see Theorem 4.3 in [PSZ1]), but no proof has been given so far.

9.2.4 The Case $\mathfrak{k} \simeq \mathfrak{sl}(2)$

The existence of the reconstruction procedure from Sect. 9.2.2 suggests that there may be a close relationship between the parabolic category $\mathcal{O}$ for a minimal compatible parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a certain subcategory of admissible $(\mathfrak{g}, \mathfrak{k})$-modules (i.e., $(\mathfrak{g}, \mathfrak{k})$-modules of finite type). This is in general not an equivalence of categories, but becomes such after a suitable modification of the parabolic category $\mathcal{O}$. The appropriate modification has been carried out in [PSZ2] for the case $\mathfrak{k} \simeq \mathfrak{sl}(2)$, and we present (without proof) the result in this section.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and $\mathfrak{k}$ be an arbitrary $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{g}$. Fix a standard $\mathfrak{sl}(2)$-basis $\{e, f, h\}$ of $\mathfrak{k}$, and note that h is a semisimple element of $\mathfrak{g}$. Let $\mathfrak{t} = \mathbb{C}h$ be the toral subalgebra of $\mathfrak{g}$ spanned by h, and let $C(\mathfrak{t})$ be the centralizer of h in $\mathfrak{g}$. We consider the following parabolic subalgebra $\mathfrak{p}$ and its opposite parabolic subalgebra $\bar{\mathfrak{p}}$:

$$\mathfrak{p} := C(\mathfrak{t}) \ni \left(\bigoplus_{\alpha(h)>0} \mathfrak{g}^\alpha \right), \quad \bar{\mathfrak{p}} = C(\mathfrak{t}) \ni \left(\bigoplus_{\alpha(h)<0} \mathfrak{g}^\alpha \right).$$

Clearly $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are minimal compatible parabolic subalgebras for any nonzero weight $\mu \in \mathfrak{t}^*$ (see Sect. 9.2.1).

Define $C_{\bar{\mathfrak{p}},\mathfrak{t}}$ to be the category of finite-length $\mathfrak{g}$-modules that are $\bar{\mathfrak{p}}$-locally finite, $\mathfrak{t}$-semisimple and $\mathfrak{t}$-integral (i.e., h acts with integer eigenvalues). In particular, the

$\mathfrak{k}$-finite part of the parabolic category $\mathcal{O}$ for $\mathfrak{g}$ determined by $\mathfrak{p}$ is a full subcategory of the category $C_{\bar{\mathfrak{p}},\mathfrak{t}}$. By $C_{\bar{\mathfrak{p}},\mathfrak{t},n}$ for $n \in \mathbb{Z}_{\geq 0}$, we denote the n-truncated category $C_{\bar{\mathfrak{p}},\mathfrak{t}}$, i.e., the full subcategory of $C_{\bar{\mathfrak{p}},\mathfrak{t}}$ consisting of objects all $\mathfrak{t}$-weights μ of that satisfy $\mu(h) \geq n$. We also assign an integer $\Lambda := \frac{1}{2}(\lambda_1 + \lambda_2)$ to the pair $(\mathfrak{g}, \mathfrak{k})$, where λ_1 (respectively, λ_2) is the maximum (resp., submaximum) weight of $\mathfrak{t}$ in $\mathfrak{g}/\mathfrak{k}$. In this subsection, we identify $\mathfrak{t}$-weights with integers via the correspondence $\mu \rightsquigarrow \mu(h)$.

Denote by $C_{\mathfrak{k}}$ the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules of finite length. By $C_{\mathfrak{k},n}$ for $n \in \mathbb{Z}_{\geq 0}$, we denote the full subcategory of $C_{\mathfrak{k}}$ consisting of $\mathfrak{g}$-modules M such that $\mathrm{Hom}_{\mathfrak{k}}(L, M) \neq 0$ implies $\dim L > n$.

We now consider two functors: Γ^j and B_j. The functor $\Gamma_{\mathfrak{k},\mathfrak{t}}$ was defined in (9.1) in Sect. 9.1. In what follows, we set $\Gamma := \Gamma_{\mathfrak{k},\mathfrak{t}}$ and denote the right derived functors (Zuckerman functors) $R^j\Gamma_{\mathfrak{k},\mathfrak{t}}$ by Γ^j. For each j, the restriction of Γ^j to $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ is a well-defined functor from $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ to $C_{\mathfrak{k},n}$ (see [PZ5]), which we also denote by Γ^j.

Next we define a functor

$$(\mathrm{B}^{\mathfrak{k},\mathfrak{t}})^{\theta} : (\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}\text{-mod} \rightsquigarrow C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}\,,$$

where $(\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod stands for the category of finitely generated $(\mathfrak{g}, \mathfrak{t})$-modules and $\theta : Z_{U(\mathfrak{g})} \to \mathbb{C}$ denotes a fixed central character. If M is a $\mathfrak{g}$-module, then M^{θ} stands for the vectors in M on which $z - \theta(z)$ acts locally nilpotently for any $z \in Z_{U(\mathfrak{g})}$. By $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ for a variable $\ell \in \mathbb{Z}_{\geq 0}$, we denote the subcategory of $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ consisting of modules M with $M = M^{\theta}$ and such that $\mathfrak{h}$ acts via Jordan blocks of size at most ℓ. We note that $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ is a finite-length category that has an injective cogenerator $I^{\theta,\ell}_{n+2}$.

Following [PSZ2], we set

$$\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X := X \Big/ \left(\bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}\left(X,\, I^{\theta,\ell}_{n+2}\right)} \ker \varphi \right)$$

for $X \in (\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod. Then we have $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X \in C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, which shows that $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell}$ is the "largest quotient" of X lying in $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

It is essential to notice that there is a canonical surjective homomorphism

$$\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X \to \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell-1} X\,,$$

i.e., that $\left\{\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X\right\}$ is an inverse system of $\bar{\mathfrak{p}}$-locally finite $(\mathfrak{g}, \mathfrak{t})$-modules. We set

$$\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta} X := \varprojlim \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X\,.$$

It is easy to see that $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}$ is a right-exact functor from $(\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod to $\mathfrak{g}$-mod, and we denote by $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j}$ its left derived functors, that is $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j} X = L_j \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta} X$ for $X \in (\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod.

Let $C^{\theta}_{\mathfrak{k},n}$ and $C^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ be the respective subcategories of $C_{\mathfrak{k},n}$ and $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ consisting of $\mathfrak{g}$-modules M with $M = M^{\theta}$, Corollary 18 in [PSZ2] states that in fact $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j}$ is a well-defined functor from $C^{\theta}_{\mathfrak{k},n}$ to $C^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Since $\mathfrak{k}$ and $\mathfrak{t}$ are fixed, we may set $\mathrm{B}^{\theta,\ell} := \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell}$, $\mathrm{B}^{\theta} := \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}$, and $\mathrm{B}^{\theta}_{j} := \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j}$. By the same letters we also denote the restrictions of these functors to the category $C^{\theta}_{\mathfrak{k},n}$. We call the functor $\mathrm{B}_j := \bigoplus_{\theta} \mathrm{B}^{\theta}_{j}$ the jth-*Serganova functor*.

The main result from [PSZ2] is the following.

Theorem 9.18 *For any $n \geq \Lambda$, the* 1*st-Zuckerman functor*

$$\Gamma^1 : C_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow C_{\mathfrak{k},n}$$

and the 1*st-Serganova functor*

$$\mathrm{B}_1 : C_{\mathfrak{k},n} \rightsquigarrow C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$$

are mutually inverse equivalences of categories.

In [PSZ2], also some applications of Theorem 9.18 are indicated. We conclude this section by noting that for the Lorentz pair $\left(\mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \text{diagonal}\,\mathfrak{sl}(2)\right)$, a description of the category $C_{\mathfrak{k}}$ was given by I. Gelfand and V. Ponomarev in [GelP] already in 1967.

9.3 The Case when $\mathfrak{g}$ is Infinite Dimensional

We can finally turn our attention to the general case of an infinite-dimensional Lie algebra $\mathfrak{g}$. Let's assume for concreteness that $\mathfrak{g}$ is locally reductive, $\mathfrak{g} = \bigcup_i \mathfrak{g}_i$, and let $\mathfrak{k}$ be the union $\bigcup_i \mathfrak{k}_i$ of reductive in $\mathfrak{g}_i$ subalgebras $\mathfrak{k}_i \subset \mathfrak{g}_i$. Then we can ask the same question as in Sect. 9.2.1, namely: does there exist an infinite-dimensional simple $(\mathfrak{g},\mathfrak{k})$-module of finite type in this setting? (Note that we no longer assume $\mathfrak{k}$ to act semisimply when we say $(\mathfrak{g},\mathfrak{k})$-module.) In almost all cases these questions are wide open.

In fact, one may wish to state the question a bit more carefully in this case by asking that the $(\mathfrak{g},\mathfrak{k})$-module in question be non-integrable, as integrable modules are analogs of finite-dimensional modules. In Theorem 9.22, we will answer this question positively for a special class of subalgebras $\mathfrak{k}$.

Before we do this, let us look at an example of an integrable $(\mathfrak{g},\mathfrak{k})$-module of finite type that is very easy to construct. We start with an important preliminary

proposition. Its proof is presented in [PZ4] and is based on earlier results from [HTW].

Proposition 9.19 *Let $\mathfrak{s}_n$ be a sequence of classical finite-dimensional simple Lie algebras of rank n and of fixed type A, B, C or D. Denote by V_n the natural $\mathfrak{s}_n$-module. Then for any fixed $a, b, c, k \in \mathbb{Z}_{>0}$ the length of the $\mathfrak{s}_n$-module $T^k(V_n^a \oplus (V_n^*)^b \oplus \mathbb{C}^c)$ stabilizes when $n \to \infty$ (here $\mathbb{C}$ stands for the trivial one-dimensional $\mathfrak{s}_n$-module).*

Recall the definition of the diagonal Lie algebra $\mathfrak{sl}(i_0 \prod_i \theta_i)$ from Sect. 4.2. In the present chapter, we set $p := i_0$, $\Theta = \prod_i \theta_i$, where $\theta_1, \theta_2, \ldots$ is an infinite sequence of elements of $\mathbb{Z}_{>0}$, and write $p\Theta$ instead of $i_0 \prod_i \theta_i$. We consider $\mathfrak{g} = \mathfrak{gl}(p\Theta)$ instead of $\mathfrak{sl}(p\Theta)$, i.e., the union of the chain of diagonal inclusions

$$\mathfrak{gl}(p) \subset \mathfrak{gl}(p\theta_1) \subset \mathfrak{gl}(p\theta_1\theta_2) \subset \cdots$$

where, for $n \geq 0$, $\mathfrak{gl}(p\theta_1\theta_2 \cdots \theta_n)$ is embedded into $\mathfrak{gl}(p\theta_1 \cdots \theta_{n+1})$ by placing a matrix $A \in \mathfrak{gl}(p\theta_1 \cdots \theta_n)$ repeatedly θ_{n+1} times along the main diagonal in $\mathfrak{gl}(p\theta_1 \cdots \theta_{n+1})$. The reader will verify immediately that $\mathfrak{gl}(p\Theta) = Z_{\mathfrak{gl}(p\Theta)} \oplus \mathfrak{sl}(p\Theta)$, where the center $Z_{\mathfrak{gl}(p\Theta)}$ is one-dimensional.

We define $\mathfrak{k}$ as follows. Let $\mathfrak{k}_0 := \mathfrak{g}_1 = \mathfrak{gl}(p)$. Set $\mathfrak{k}_n := \mathfrak{k}_0 + C_{\mathfrak{g}_n}(\mathfrak{k}_0)$ where $\mathfrak{g}_n = \mathfrak{gl}(p\theta_1 \cdots \theta_{n-1})$, and let $\mathfrak{k} := \bigcup_{n \in \mathbb{Z}_{>0}} \mathfrak{k}_n$. Then, as it is easy to check, we have $C_{\mathfrak{g}_n}(\mathfrak{k}_0) = \mathfrak{gl}(\theta_1 \cdots \theta_{n-1})$, and the inclusion $C_{\mathfrak{g}_n}(\mathfrak{k}_0) \subset C_{\mathfrak{g}_{n+1}}(\mathfrak{k}_0)$ is nothing but the θ_{n+1}-diagonal inclusion. Hence $\mathfrak{k} \simeq \mathfrak{gl}(p) \oplus \mathfrak{gl}(\Theta)$.

Proposition 9.20 *The adjoint representation of $\mathfrak{gl}(p\Theta)$ is a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module of finite length and thus, in particular, is a $(\mathfrak{gl}(p\Theta), \mathfrak{k})$-module of finite type.*

Proof The statement follows from the observation that for each n, the adjoint representation of $\mathfrak{gl}(p\theta_1 \cdots \theta_{n-1})$ considered as a $C_{\mathfrak{g}_n}(\mathfrak{k}_0) = \mathfrak{gl}(\theta_1 \cdots \theta_{n-1})$-module is a submodule of $T^2(V_n^p \oplus (V_n^*)^p)$, where V_n is the natural $\mathfrak{gl}(\theta_1 \cdots \theta_n)$-module. By Proposition 9.19, the length of $T^2(V_n^p \oplus (V_n^*)^p)$ as an $\mathfrak{sl}(\theta_1 \cdots \theta_{n-1})$-module stabilizes for $n \to \infty$, hence the length of $\mathfrak{gl}(p\Theta_1 \cdots \Theta_n)$ considered as a $C_{\mathfrak{g}_n}(\mathfrak{k}_0)$-module is bounded for $n \to \infty$. The reader will check immediately that this implies that the adjoint module of $\mathfrak{gl}(p\Theta)$ has finite length as a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. □

Now we outline a construction from [PZ4] of a generalization of the fundamental series to a locally reductive Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ and let $\mathfrak{k}_0 \subset \mathfrak{g}_1$ be a finite-dimensional subalgebra reductive in $\mathfrak{g}$ (equivalently, in $\mathfrak{g}_1$). Fix a Cartan subalgebra $\mathfrak{t}_0$ in $\mathfrak{k}_0$. For any $\mathfrak{g}_n$, we have the notion of a $\mathfrak{t}_0$-*compatible parabolic subalgebra* of $\mathfrak{g}_n$: by definition this is a parabolic subalgebra $\mathfrak{p}_n \subset \mathfrak{g}_n$ of the form $\bigoplus_{\sigma, \operatorname{Re}\sigma \geq 0} (\mathfrak{g}_n)^{\sigma}_{h_n}$, where h_n is a semisimple element of $\mathfrak{t}_0$, the variable σ runs over the eigenvalues of h_n in $\mathfrak{g}_n$, and $(\mathfrak{g}_n)^{\sigma}_{h_n}$ are the corresponding eigenspaces. We call a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ a $\mathfrak{t}_0$-*compatible parabolic subalgebra* if, for all n, $\mathfrak{p} \cap \mathfrak{g}_n$ is a $\mathfrak{t}_0$-compatible parabolic subalgebra of $\mathfrak{g}_n$ and $\mathfrak{n}_n = \mathfrak{n}_{n+1} \cap \mathfrak{g}_n$, where $\mathfrak{n}_n$ is the nilradical of $\mathfrak{p}_n$. It is possible (but not required) that there is a semisimple element $h \in \mathfrak{t}_0$ such that $\mathfrak{p} = \bigoplus_{\sigma, \operatorname{Re}\sigma \geq 0} \mathfrak{g}^{\sigma}_h$.

One can always choose decompositions $\mathfrak{p}_n = \mathfrak{m}_n \supset \mathfrak{n}_n$ where, for each n, $\mathfrak{m}_n$ is a reductive in $\mathfrak{g}_n$ subalgebra such that $\mathfrak{m}_{n+1} \cap \mathfrak{g}_n = \mathfrak{m}_n$. This yields a decomposition $\mathfrak{p} = \mathfrak{m} \supset \mathfrak{n}$, where $\mathfrak{m} = \bigcup_n \mathfrak{m}_n$ and $\bigcup_n \mathfrak{n}_n$. By definition, $\mathfrak{n}$ is the *nilradical* of $\mathfrak{p}$ and $\mathfrak{m}$ is a locally reductive subalgebra of $\mathfrak{g}$. In what follows, we consider the decomposition $\mathfrak{p} = \mathfrak{m} \supset \mathfrak{n}$ fixed and define $\bar{\mathfrak{n}}$ as the union $\bigcup_n \bar{\mathfrak{n}}_n$, where for each n, $\mathfrak{g}_n = \bar{\mathfrak{n}}_n \oplus \mathfrak{m}_n \oplus \mathfrak{n}_n$ is the canonical $\mathfrak{m}_n$-module decomposition. In this way, $\bar{\mathfrak{n}}$ is of course an integrable $\mathfrak{m}$-module.

Let $\mathfrak{k} := \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$. Then $\mathfrak{k}_n = \mathfrak{k}_0 + C_{\mathfrak{g}_n}(\mathfrak{k}_0)$ is reductive in $\mathfrak{g}$ for each n. Note that $\mathfrak{k} \cap \mathfrak{m} = \mathfrak{m}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$, where $\mathfrak{m}_0 := \mathfrak{k}_0 \cap \mathfrak{m}$. Our goal is to construct nontrivial $(\mathfrak{g}, \mathfrak{k})$-modules by starting with a nontrivial $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module E and then applying a Zuckerman functor.

We first extend E to a $\mathfrak{p}$-module by setting $\mathfrak{n} \cdot E = 0$. We then consider the induced module $M(\mathfrak{p}, E) := \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} E$. This is an integrable $\mathfrak{m} \cap \mathfrak{k}$-module. Indeed, the equality of $\mathfrak{m}$-modules $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$ implies via the Poincaré–Birkhoff–Witt Theorem that $M(\mathfrak{p}, E)$ has an $\mathfrak{m}$-module filtration with associated graded module $\mathrm{Gr}\, M(\mathfrak{p}, E)$ equal to $S^{\cdot}(\bar{\mathfrak{n}}) \otimes E$. Both $S^{\cdot}(\bar{\mathfrak{n}})$ and E are integrable $\mathfrak{m} \cap \mathfrak{k}$-modules, thus $M(\mathfrak{p}, E)$ is also $(\mathfrak{m} \cap \mathfrak{k})$-integrable.

We now set $A(\mathfrak{p}, E) := R^s \Gamma_{\mathfrak{k}_0, \mathfrak{m}_0}(M(\mathfrak{p}, E))$, where $R^s \Gamma_{\mathfrak{k}_0, \mathfrak{m}_0}$ is the sth-Zuckerman functor with $s := \frac{1}{2} \dim(\mathfrak{k}_0/\mathfrak{m}_0)$. By definition $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k}_0)$-module with semisimple action of $\mathfrak{k}$, but in fact $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module (see Theorem 9.22 below). We also set $A(\mathfrak{p}_0, E) := R^s \Gamma_{\mathfrak{k}_0, \mathfrak{m}_0}(\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} E)$, where $\mathfrak{p}_0 := \mathfrak{k}_0 \cap \mathfrak{p}$ and we regard E as a module over $\mathfrak{m}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$ and $\mathrm{ind}_{\mathfrak{p}_0}^{\mathfrak{k}_0} E$ as a $\mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$ module. It is easy to see that there is a functorial morphism of $\mathfrak{k}_0$-modules

$$\Psi_E : A(\mathfrak{p}_0, E) \to A(\mathfrak{p}, E).$$

Knapp and Vogan [KnVo] call Ψ_E the *bottom layer map*. Following [PZ4], we call any $\mathfrak{g}$-subquotient of $A(\mathfrak{p}, E)$ generated by vectors in $\mathrm{im}\Psi_E$ a *bottom layer subquotient* of $A(\mathfrak{p}, E)$.

Note that $\mathfrak{m}_0 \cap C_{\mathfrak{g}}(\mathfrak{k}_0) = Z_{\mathfrak{k}_0}$. Therefore if $\mathfrak{b}_{\mathfrak{m}_0}$ is a fixed Borel subalgebra of $\mathfrak{m}_0$, we can decompose E as

$$\bigoplus_{\nu} V_{\mathfrak{m}_0}(\nu) \boxtimes_{U(Z_{\mathfrak{k}_0})} E''_{\nu},$$

where we consider $E''_{\nu} := \mathrm{Hom}_{\mathfrak{m}_0}(V_{\mathfrak{m}_0}(\nu), E)$ as a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module and ν runs over all $\mathfrak{b}_{\mathfrak{m}_0}$-dominant integral weights of $\mathfrak{m}_0$.

Now fix a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{k}_0$ such that $\mathfrak{b}_0 \cap \mathfrak{m}_0 = \mathfrak{b}_{\mathfrak{m}_0}$. This defines two Weyl group elements: the element $w_{\mathfrak{k}_0} \in W_{\mathfrak{k}_0}$ of maximal length with respect to $\mathfrak{b}_0$, and the element $w_{\mathfrak{m}_0} \in W_{\mathfrak{m}_0}$ of maximal length with respect to $\mathfrak{b}_0 \cap \mathfrak{m}_0$. For any $\mathfrak{b}_{\mathfrak{m}_0}$-dominant $\mathfrak{k}_0$-integral weight ν, we set

$$\nu^{\vee} := w_{\mathfrak{k}_0} \circ w_{\mathfrak{m}_0}^{-1}(\nu + \rho_{\mathfrak{b}_0}) - \rho_{\mathfrak{b}_0},$$

where $\rho_{\mathfrak{b}_0}$ is the half-sum of the $\mathfrak{b}_0$-positive roots of $\mathfrak{k}_0$.

Lemma 9.21 *The $\mathfrak{k}$-module $A(\mathfrak{p}_0, E)$ is $\mathfrak{k}$-integrable and is isomorphic to*

$$\bigoplus_{\nu} V_{\mathfrak{k}_0}(\nu^\vee) \boxtimes_{U(Z_{\mathfrak{k}_0})} E''_\nu,$$

where as above ν runs over all dominant integral weights of $\mathfrak{m}_0$, and where $V_{\mathfrak{k}_0}(\nu^\vee) := 0$ whenever $\nu^\vee$ is not $\mathfrak{b}_0$-dominant and integral for $\mathfrak{k}_0$.

Proof This statement is a direct corollary of Proposition 6.3 in [EW], which is an analog of the Bott–Borel–Weil Theorem. □

The main result of this section is the following theorem from [PZ4] about constructing $(\mathfrak{g}, \mathfrak{k})$-modules of finite type for a locally reductive Lie algebra $\mathfrak{g}$. For its proof we refer the reader to [PZ4].

Theorem 9.22 *The following statements hold.*

(1) $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module.

(2) If $M(\mathfrak{p}, E)$ is an $(\mathfrak{m}, \mathfrak{k}\cap\mathfrak{m})$-module of finite type, then $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type.

(3) Assume $E = \bigcup_n E_n$ where each E_n is an $(\mathfrak{m}_n, \mathfrak{k} \cap \mathfrak{m}_n)$-module on which $Z_{\mathfrak{m}_n}$ acts via a one-dimensional representation. Then the bottom layer map Ψ_E is an injection. Assume that for some ν, $E''_\nu \neq 0$ and $\nu^\vee$ is dominant integral for $\mathfrak{k}_0$. Then $\mathrm{Hom}_{\mathfrak{k}_0}(L_{\mathfrak{k}_0}(\nu^\vee), A(\mathfrak{p}, E)) = E''_\nu$. Hence $A(\mathfrak{p}, E)$ has a simple bottom layer subquotient.

(4) Assume $E = \bigcup_n E_n$ where each E_n is an $(\mathfrak{m}_n, \mathfrak{k} \cap \mathfrak{m}_n)$-module with $Z_{U(\mathfrak{m}_n)}$-character, and that E satisfies: $A(\mathfrak{p}, E) \neq 0$ and for some N, the $Z_{U(\mathfrak{g}_N)}$-character of $\mathrm{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N} E_N$ is not regular integral. Then some bottom layer subquotient of $A(\mathfrak{p}, E)$ is not an integrable $\mathfrak{g}$-module. If, in addition, $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$, then some simple bottom layer subquotient of $A(\mathfrak{p}, E)$ is a strict $(\mathfrak{g}, \mathfrak{k})$-module.

(5) Under the assumptions of (3) assume further that $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{t}_0)$ and that E is simple. Then $\mathfrak{t}_0$ acts via weight $\mu \in \mathfrak{t}_0^*$ on E, $\mu^\vee$ is a dominant integral weight for $\mathfrak{k}_0$, and there is an isomorphism of $\mathfrak{k} = \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$-modules

$$A(\mathfrak{p}_0, E) \simeq L_{\mathfrak{k}_0}(\mu^\vee) \boxtimes_{U(Z_{\mathfrak{k}_0})} E''.$$

where E'' equals E considered as a $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. Furthermore, Ψ_E yields an isomorphism between the $\mathfrak{k}$-modules $L_{\mathfrak{k}_0}(\mu^\vee) \otimes \mathrm{Hom}_{\mathfrak{k}_0}(L_{\mathfrak{k}_0}(\mu^\vee), A(\mathfrak{p}, E))$ and $A(\mathfrak{p}_0, E)$.

(6) If, under the assumptions of (5), $\mathrm{im}\,\Psi_E$ is a simple $\mathfrak{k}$-submodule of $A(\mathfrak{p}, E)$, then $A(\mathfrak{p}, E)$ has a unique simple bottom layer subquotient. A sufficient condition for the simplicity of $\mathrm{im}\,\Psi_E$ is the inclusion $\mathfrak{m} \subset \mathfrak{k}$.

In the remainder of this chapter, we illustrate Theorem 9.22 in the specific case of the diagonal Lie algebra $\mathfrak{g} = \mathfrak{gl}(p\Theta)$. The $(\mathfrak{g}, \mathfrak{k})$-modules $A(\mathfrak{p}, E)$ in Theorem 9.22*(1)* form a series that generalizes the fundamental series from

Sect. 9.2.1 to a locally reductive Lie algebra $\mathfrak{g}$ for subalgebras $\mathfrak{k} \subset \mathfrak{g}$ of the form $\mathfrak{k} = \mathfrak{k}_0 + C_{\mathfrak{g}}(\mathfrak{k}_0)$. Part *(2)* from Theorem 9.22 shows that the $(\mathfrak{g}, \mathfrak{k})$-module $A(\mathfrak{p}, E)$ has finite type if $M(\mathfrak{p}, E)$ is an $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module of finite type. Our first step in the particular case $\mathfrak{g} = \mathfrak{gl}(p\Theta)$ is to construct "input data", namely, to construct modules $M(\mathfrak{p}, E)$ that are $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-modules of finite type, see Proposition 9.27 below.

Fix the exhaustion $\mathfrak{g} = \bigcup_n \mathfrak{gl}(p\theta_1 \cdots \theta_{n-1})$ as above. Let $\mathfrak{k}_0 \subset \mathfrak{g}_1 = \mathfrak{gl}(p)$ be any reductive in $\mathfrak{g}_1$ subalgebra that contains a $\mathfrak{g}_1$-regular element h, and such that the p-dimensional natural $\mathfrak{gl}(p)$-module $\mathbb{C}^p$ is simple as a $\mathfrak{k}_0$-module. For instance, $\mathfrak{k}_0$ may equal $\mathfrak{gl}(p)$, $\mathfrak{sl}(p)$ or a principal $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{sl}(p)$. Let $\mathfrak{t}_0 := C_{\mathfrak{k}_0}(h)$. We define $\mathfrak{p}$ as the $\mathfrak{t}_0$-compatible parabolic subalgebra $\bigoplus_{\sigma, \operatorname{Re}\sigma \geq 0} \mathfrak{g}_h^\sigma$.

Lemma 9.23

(1) $\mathfrak{m} \cap \mathfrak{g}_n \simeq \mathfrak{gl}(\theta_1 \cdots \theta_{n-1})^p$.
(2) $C_{\mathfrak{g}_n}(\mathfrak{k}_0) \simeq \mathfrak{gl}(\theta_1 \cdots \theta_{n-1})$ *is the diagonal subalgebra in* $\mathfrak{gl}(\theta_1 \cdots \theta_{n-1})^p$.

Proof As a $C_{\mathfrak{g}_n}(h)$-module, the natural representation V_n of $\mathfrak{gl}(p\theta_1 \cdots \theta_{n-1})$ decomposes as a direct sum of p isotypic components each of dimension $\theta_1 \cdots \theta_{n-1}$. This yields *(1)*.

As a $\mathfrak{k}_0$-module V_n decomposes as a direct sum of $\theta_1 \cdots \theta_{n-1}$ copies of the simple $\mathfrak{k}_0$-module $\mathbb{C}^p$. This implies *(2)*. □

Corollary 9.24

(1) $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{t}_0) = \mathfrak{gl}(\Theta)^p$;
(2) $\mathfrak{k} \simeq \mathfrak{k}_0 + \mathfrak{gl}(\Theta)$, $\mathfrak{k}_0 \cap \mathfrak{gl}(\Theta) \subset Z_{\mathfrak{gl}(\Theta)}$;
(3) if $\mathfrak{k}_0 = \mathfrak{gl}(p)$, *then* $\mathfrak{k} \simeq \mathfrak{gl}(p) + \mathfrak{gl}(\Theta)$ *is a maximal proper subalgebra of* $\mathfrak{gl}(p\Theta)$.

We now construct a class of simple $\mathfrak{gl}(\Theta)$-modules. Let V_n denote the natural representation of $\mathfrak{gl}(\theta_1 \cdots \theta_{n-1})$. Fix $n_0 > 1$ and let $L(\lambda_{n_0})$ be the simple finite-dimensional $\mathfrak{gl}(\theta_1 \cdots \theta_{n_0-1})$-module with highest weight $\lambda_{n_0} = (\lambda^1, \ldots, \lambda^{\theta_1 \cdots \theta_{n_0-1}})$, $\lambda^i \geq \lambda^{i+1}$. Define $n' = n'(\lambda_{n_0-1})$ as the largest index for which the entry $\lambda^{n'}$ is non-negative; if $\lambda^1 < 0$, we set $n' = 0$. To λ_{n_0} we assign the following highest weight of $\mathfrak{gl}(\theta_1 \cdots \theta_{n_0})$:

$$\lambda_{n_0+1} := (\lambda^1, \ldots, \lambda^{n'}, \underbrace{0, 0, \ldots, 0,}_{\theta_1 \cdots \theta_{n_0}(\theta_{n_0+1}-1)\text{times}} \lambda^{n'+1}, \ldots, \lambda^{\theta_1 \cdots \theta_{n_0}}).$$

Lemma 9.25 *There is a natural injection of* $\mathfrak{gl}(\theta_1 \cdots \theta_{n_0-1})^{\theta_{n_0}}$*-modules*

$$L(\lambda_{n_0})^{\theta_{n_0}} \to L(\lambda_{n_0+1}),$$

and hence a diagonal injection of $\mathfrak{gl}(\theta_1 \cdots \theta_{n-1})$*-modules*

$$L(\lambda_n) \to L(\lambda_{n+1})$$

for any $n > n_0$.

Proof The natural injection $V_{n_0}^{\theta_{n_0}} \to V_{n_0+1}$ induces an injection of $\mathfrak{gl}(\theta_1 \cdots \theta_{n_0-1})^{\theta_{n_0}}$-modules

$$T^{\cdot}(V_{n_0} \oplus V_{n_0}^*)^{\theta_{n_0}} \to T^{\cdot}(V_{n_0+1} \oplus V_{n_0+1}^*),$$

which in turn induces an injection

$$L(\lambda_{n_0})^{\theta_{n_0}} \to L(\lambda_{n_0+1}),$$

as required. □

Corollary 9.26 *For every* $n_0 > 1$ *and for any dominant integral weight* λ_{n_0} *of* $\mathfrak{gl}(\theta_1 \cdots \theta_{n_0-1})$, $\tilde{L}(\lambda_{n_0})$ *is a simple* $\mathfrak{gl}(\Theta)$*-module defined as the direct limit* $\varinjlim_{n \geq n_0} L(\lambda_n)$, *where* $L(\lambda_n)$ *is embedded diagonally into* $L(\lambda_{n+1})$ *according to Lemma 9.25.*

Now let $\lambda_{n_0^1} \cdots \lambda_{n_0^p}$ be p dominant weights as in Corollary 9.26. Assume that the ordering of the weights is compatible with $\mathfrak{n}$, i.e., that the h value of any root $\varepsilon_i - \varepsilon_j$, $i < j$, of $\mathfrak{g}_1 = \mathfrak{gl}(p)$ has a non-negative real part. Define E as $L(\lambda_{n_0^1}) \boxtimes \cdots \boxtimes \tilde{L}(\lambda_{n_0^p})$ with trivial action of $\mathfrak{n}$.

Proposition 9.27 $M(\mathfrak{p}, E) = \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} E$ *is an* $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$*-module of finite type.*

Proof It suffices to show that $\mathrm{Gr}\, M(\mathfrak{p}, E)$ is an $(\mathfrak{m}, \mathfrak{k} \cap \mathfrak{m})$-module of finite type. As a $\mathfrak{m}$-module, $\mathrm{Gr}\, M(\mathfrak{p}, E)$ is isomorphic to $S^{\cdot}(\bar{\mathfrak{n}}) \otimes E$, and is in particular a weight module over the Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. This subalgebra acts via a single weight on E and via arbitrary sums of $\mathfrak{p}$-negative $\mathfrak{t}_0$-weights on $S^{\cdot}(\bar{\mathfrak{n}})$. Since each $\mathfrak{t}_0$-weight of $S^{\cdot}(\bar{\mathfrak{n}})$ occurs only in finitely many symmetric powers of $\bar{\mathfrak{n}}$, it suffices to show that each fixed tensor product $S^t(\bar{\mathfrak{n}}) \otimes E$ is a $\mathfrak{k} \cap \mathfrak{m}$-module of finite length. Notice that E is a direct limit $\varinjlim_{n \geq \max(n_0^1, \ldots n_0^p)} E_n$, such that each E_n is a $C_{\mathfrak{g}_n}(\mathfrak{k}_0) \simeq \mathfrak{gl}(\theta_1 \cdots \theta_{n-1})$-submodule of a fixed tensor power $T^k(V_n^p \oplus (V_n^*)^p)$. Hence $S^t(\bar{\mathfrak{n}}_n) \otimes E_n$ is also contained in a fixed tensor power $T^k(V_n^p \oplus (V_n^*)^p)$. Proposition 9.19 now implies that for each n, $S^t(\bar{\mathfrak{n}}_n) \otimes E_n$ is a $C_{\mathfrak{g}}(\mathfrak{k}_0) \cap \mathfrak{g}_n$-module of finite length, hence $S^t(\bar{\mathfrak{n}}) \otimes E$ is a $\mathfrak{k} \cap \mathfrak{m}$-module of finite length. The proposition follows. □

It will follow from Proposition 9.27 and Theorem 9.22*(2)* that $A(\mathfrak{p}, E)$ is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type. However, we must first ensure that $A(\mathfrak{p}, E)$ is nonzero. For this, it suffices, by Theorem 9.22*(5)*, to ensure that the weight $\mu^\vee$ is integral $\mathfrak{k}_0$-dominant. An easy computation shows that the weight μ is nothing but the weight

$(\sum_i \lambda^i_{n_0^1}, \sum_i \lambda^i_{n_0^2}, \dots, \sum_i \lambda^i_{n_0^p})$ of $\mathfrak{g}_1$, restricted to $\mathfrak{t}_0$. Let $\mathfrak{k}_0 = \mathfrak{gl}(p)$. So the regularity and $\mathfrak{k}_0$-dominancy condition on $\mu^\vee$ are equivalent to the condition

$$\sum_i \lambda^i_{n_0^1} \le \sum_i \lambda^i_{n_0^2} \le \cdots \le \sum_i \lambda^i_{n_0^p}.$$

Note furthermore that our choice of weights $\lambda_{n_0^1}, \dots, \lambda_{n_0^p}$ allows for the possibility that the $Z_{U(\mathfrak{g}_N)}$-character of $\mathrm{ind}_{\mathfrak{p}_N}^{\mathfrak{g}_N} E_N$ be non-regular for some N, and hence in the latter case, no irreducible bottom layer quotient of $A(\mathfrak{p}, E)$ is $\mathfrak{g}$-integrable. Since $\mathfrak{k}_0 = \mathfrak{gl}(p)$, $\mathfrak{k}$ is a maximal proper subalgebra of $\mathfrak{gl}(p\Theta)$. This implies (via Theorem 9.22*(6)*) that whenever $A(\mathfrak{p}, E)$ is not integrable, any irreducible bottom layer quotient of $A(\mathfrak{p}, E)$ is a strict $(\mathfrak{g}, \mathfrak{k})$-module. In particular, $\mathfrak{k} = \mathfrak{gl}(p) + \mathfrak{gl}(\Theta)$ is a primal subalgebra of $\mathfrak{gl}(p\Theta)$.

Finally, Lemma 9.23 implies that the condition $\mathfrak{m} \subset \mathfrak{k}$ from Theorem 9.22*(6)* holds only when $p = 1$. However in this case $s = 0$, hence the claim of *(6)* is trivial. Nevertheless, there is an interesting nontrivial case in which Theorem 9.22*(6)* applies: this is when $\lambda_{n_0^1} = \cdots = \lambda_{n_0^{p-1}} = 0$ and $\lambda_{n_0^p} \neq 0$. In this latter case E'' is clearly a simple $C_{\mathfrak{g}}(\mathfrak{k}_0)$-module. Furthermore, as it is easy to see, for large n the $Z_{U(\mathfrak{g}_n)}$-character of $\mathrm{ind}_{\mathfrak{p}_n}^{\mathfrak{g}_n} E_n$ is integral but not regular, hence the $(\mathfrak{g}, \mathfrak{k})$-module $A(\mathfrak{p}, E)$ has a unique strict simple subquotient.

9.4 Closing Remarks

We now conclude this chapter with a few remarks.

1. We did not discuss applications of Theorem 9.14 in the case of a root-reductive Lie algebra $\mathfrak{g}$, since this would make a long chapter even longer. For a root-reductive Lie algebra, Theorem 9.14 constructs interesting $(\mathfrak{g}, \mathfrak{k})$-module s that certainly deserve further study. For a discussion of these modules, see Section 5 in [PZ4].
2. The following natural question arises: what is an analog of the first reconstruction theorem (Theorem 9.8) for an infinite-dimensional $\mathfrak{g}$? We do not know the answer, since the genericity condition "gets lost at infinity". In other words, if $L(\mu)$ is a $\mathfrak{k}$-type, $\mu = \varprojlim \mu_n$, then for large N, we cannot guarantee that $L(\mu_N)$ is a generic $\mathfrak{k}_N$-type. (Check this in examples!) Therefore we do not know whether one can prove an analog of Theorem 9.8 in the infinite-dimensional case.
3. Finding analogs of Theorems 9.7, 9.14, 9.18 in the infinite-dimensional setting are wide open problems.

4. Let $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ and let $\mathfrak{k}$ be a symmetric subalgebra (i.e., let $\mathfrak{k}$ equal the fixed points of an involution of $\mathfrak{g}$) that is not of the form considered in Theorem 9.22. For instance, if $\mathfrak{g} = \mathfrak{sl}(\infty)$, then the reader can check that $\mathfrak{k}$ could be isomorphic to $(\mathfrak{sl}(\infty) \oplus \mathfrak{sl}(\infty)) \oplus \mathbb{C}$. Does there exist an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type? Can one say something about the case when $\mathfrak{k}$ is not symmetric?

Part III
Geometric Aspects

In this final part of the book, we establish a connection of the representation theory of Lie groups and Lie algebras to geometry. We start with a review of the famous Bott–Borel–Weil Theorem and recall our favorite proof of this theorem due to Michel Demazure. We then discuss an extension of Bott–Borel–Weil's Theorem to the classical Lie supergroups $GL(m|n)$, $SOSp(m|2n)$, $\mathbf{P}(n)$, $Q(n)$. After a short review of necessary preliminaries, we reproduce some work on this topic of the first author and Igor Skornyakov from the 1980s. A main difference from the case of ordinary Lie groups is that new difficulties arise when a relevant weight is not regular (or not typical) with respect to odd roots. In this special case, the Bott–Borel–Weil theory for Lie supergroups is still unfinished.

The third part of Chap. 10 is devoted to an extension of the Bott–Borel–Weil Theorem to locally reductive ind-groups. We start with the case of a line bundle where we reproduce a result of Ivan Dimitrov, the first author and Joseph A. Wolf. We then present some partial results in the case of an inverse limit of bundles of growing rank, and conclude the book with some open problems.

Chapter 10
The Bott–Borel–Weil Theorem

The previous nine chapters were pure algebra. In this chapter, we give a glimpse into the interaction between algebra and geometry in representation theory. The Bott–Borel–Weil Theorem is one of the origins of geometric representation theory, which is currently a leading branch of representation theory. From now on we assume some basic knowledge of algebraic geometry.

10.1 The Classical Bott–Borel–Weil Theorem

Let G be a *connected reductive algebraic group*. By definition, this means that G is a connected complex linear algebraic group whose Lie algebra $\mathfrak{g}$ is reductive. For instance, G may equal one of the classical complex groups $GL(n)$, $SL(n)$, $SO(n)$, $Sp(2k)$. The famous Bott–Borel–Weil Theorem establishes a fundamental bridge between finite-dimensional irreducible representations of G and cohomology of G-linearized vector bundles on the compact homogeneous spaces of G. The classical references for this theorem are [BW] and [Bo]. Here we follow the well-known algebraic proof given by M. Demazure in [De1]. See [De2] for a shorter proof also given by Demazure. For our presentation below, we prefer the proof given in [De1] as it generalizes beautifully to both Lie supergroups and ind-groups.

Recall that a *Borel subgroup* B of G is a maximal connected closed solvable algebraic subgroup of G. More generally, a *parabolic subgroup* P of G is a connected closed algebraic subgroup of G containing a Borel subgroup. It is a classical result that if Q is a connected closed algebraic subgroup of G, the homogeneous space G/Q is compact (or equivalently projective) if and only if Q is a parabolic subgroup of G.

Recall also that if U is a vector space, then the projective space $\mathbb{P}(U)$, called the *projectivization of* U, is the set of all 1-dimensional subspaces of U. We write $\mathbb{P}^r$ for $\mathbb{P}(U)$ for an unspecified vector space U of dimension $r+1$, for instance $\mathbb{C}^{r+1}$.

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7_10

10.1.1 Introductory Considerations

We now start to introduce the main players. Let us consider the case of $G = GL(n)$ in more detail. In what follows, we fix a vector space V of dimension n and consider the group $GL(V)$ of invertible linear operators on V. We identify $GL(n)$ with $GL(V)$. Each Borel subgroup of $GL(n)$ is the stabilizer of a unique flag of maximal length

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V.$$

We discussed this topic in Chap. 5 in the language of the Lie algebra $\mathfrak{gl}(n)$, and the result about the group $GL(n)$ follows immediately from the well-known canonical 1-1 correspondence between parabolics subgroups of $GL(n)$ and parabolic subalgebras of $\mathfrak{gl}(n)$. Fix a Borel subgroup $B \subset G$. Each (closed) point of the projective algebraic variety G/B corresponds to a Borel subgroup conjugate to B via the action of G on G/B, and it is a fundamental result that all Borel subgroups of G are conjugate. Therefore the points of G/B are precisely all the Borel subgroups of G, and moreover the variety G/B is identified with the variety of maximal flags in V. Under this correspondence a maximal flag $F\ell(1, 2, \ldots, n-1, V)$ in V is assigned to its stabilizer in $GL(V)$. One can think of $F\ell(1, 2, \ldots, n-1, V)$ as a "tower of projective spaces", as there is the sequence of canonical projections

$$\begin{aligned} F\ell(1, 2, \ldots, n-1, V) &\xrightarrow{\pi_1} F\ell(2, \ldots, n-1, V) \xrightarrow{\pi_2} \cdots \\ &\xrightarrow{\pi_{n-1}} F\ell(n-1, V) = \mathbb{P}(V^*), \end{aligned} \tag{10.1}$$

where $F\ell(i_1, i_2, \ldots, i_s, V)$ stands for the variety of all flags

$$V_1 \subset V_2 \subset \cdots \subset V_s \subset V$$

with $\dim V_1 = i_1$, $\dim V_2 = i_2$, …, $\dim V_s = i_s$. The variety $F\ell(n-1, V)$ is naturally identified with the projectivization $\mathbb{P}(V^*)$ of the dual space V^*, as any $(n-1)$-dimensional subspace of V determines a one-dimensional subspace of V^*, and vice versa. Furthermore, each projection π_i is a fiber bundle with fiber the projective space $\mathbb{P}^i$ (prove this!), and thus $F\ell(1, 2, \ldots, n-1, V)$ is a "tower of projective spaces" with dimensions decreasing from $n-1$ (the dimension of the base $\mathbb{P}(V^*)$) down to 1 (the dimension of the fiber of π_1).

In fact, there are many different ways to present $F\ell(1, 2, \ldots, n-1, V)$ as a "tower of projective spaces". Instead of (10.1), one can consider any chain of natural projections of the form

$$\begin{aligned} F\ell(1, 2, \ldots, n-1, V) &\xrightarrow{p_i} F\ell(1, 2, \ldots, i-1, i+1, \ldots, n-1, V) \\ &\to F\ell(1, 2, \ldots i-2, i+1, \ldots, n-1, V) \to \ldots \to \\ F\ell(i+1, \ldots, n-1, V) &\to F\ell(i+2, \ldots, n-1, V) \to \cdots \to \mathbb{P}(V^*), \end{aligned} \tag{10.2}$$

and the reader will easily show that this is also a "tower of projective spaces".

It is time to introduce the next player, the line bundles on G/P. Before we do this, let us recall some basic facts about the line bundles on a projective space $\mathbb{P}^r$ and their cohomology. By definition, a *line bundle* on any (algebraic) variety X is a vector bundle $\mathcal{L}$ of rank 1 on X. In what follows we write $\mathcal{L}^k := \mathcal{L}^{\otimes k}$ for $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{L}^0 := \mathcal{O}_X$, and $\mathcal{L}^k = (\mathcal{L}^*)^{\otimes -k}$ for $k \in \mathbb{Z}_{<0}$ where $\mathcal{L}^* := Hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is the dual bundle. The isomorphism classes of line bundles on a fixed variety X form a group under tensor multiplication of line bundles: this is the *Picard group* $\operatorname{Pic} X$ *of* X.

Describing the elements of $\operatorname{Pic} \mathbb{P}^r$ is easy. On $\mathbb{P}^r$ there is an obvious line bundle: the tautological bundle. The (geometric) fiber of this bundle at a (closed) point x of $\mathbb{P}^r$ is the one-dimensional subspace $x \subset \mathbb{C}^{r+1}$. Denote the tautological bundle by S_1. In algebraic geometry, the standard notation for S_1 is $\mathcal{O}(-1)$ as the first Chern class of S_1 equals -1. It is a classical fact that every invertible sheaf on $\mathbb{P}^r$ is isomorphic to $\mathcal{O}(\ell)$ for some $\ell \in \mathbb{Z}$, and $\operatorname{Pic} \mathbb{P}^r \simeq \mathbb{Z}$ [Har, Ch.II, Corollary 6.17].

If X is a variety and E is a vector bundle on X, by $H^s(X, E)$ we denote the s-th cohomology group of the sheaf of regular local sections of E. We write $H^{\cdot}(X, E)$ for the $\mathbb{Z}$-graded vector space $\bigoplus_s H^s(X, E)$.

The computation of the cohomology groups of the bundles S_1^k, i.e., of $H^s(\mathbb{P}(V), S_1^k)$ (or, in standard notations, $H^s(\mathbb{P}(V), \mathcal{O}(-k))$) for all $s \in \mathbb{Z}_{\geq 0}$ is an important basic result of algebraic geometry, see [Har, Ch.III, Theorem 5.1]. We will state the result in functorial form.

Proposition 10.1 *There are canonical isomorphisms*

$$\begin{gathered} H^0(\mathbb{P}(V), S_1^k) = S^{-k}(V^*) \text{ for } k \leq 0, \\ H^{n-1}(\mathbb{P}(V), S_1^k) = S^{k-n}(V) \otimes \Lambda^n(V) \text{ for } k \geq n, \\ H^s(\mathbb{P}(V), S_1^k) = 0 \text{ for other pairs } (k, s) \end{gathered} \tag{10.3}$$

where $\dim V = n$.

Theorem 5.1 in Chapter III of [Har] implies a slightly weaker result, namely that $\dim H^r(\mathbb{P}(V), S_1^k)$ is as in (10.3). The fact that the isomorphisms of (10.3) are canonical follows from the existence of the following canonical exact sequence of vector bundles on $\mathbb{P}(V)$:

$$0 \to S_1 \to V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)} \to (V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)})/S_1 \to 0.$$

Indeed, the dual exact sequence

$$0 \to ((V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)})/S_1)^* \to V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)} \to S_1^* \to 0$$

yields the canonical homomorphism

$$V^* = H^0(\mathbb{P}(V), V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}) \to H^0(\mathbb{P}(V), S_1^*),$$

which can be shown to be an isomorphism. If $k \leq 0$, the $-k$th symmetric power of the morphism

$$V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)} \to S_1^* \to 0$$

then yields an isomorphism

$$S^{-k}(V^*) = H^0(\mathbb{P}(V), S^{-k}(V^* \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)})) \to H^0(\mathbb{P}(V), S_1^k)$$

as in the top row of (10.3). Finally the isomorphism of the middle row of (10.3) is obtained from the top row by Serre duality.

The isomorphisms (10.3) are a very special case of the Bott–Borel–Weil Theorem. In fact, the theorem is first stated and proved for the variety of maximal flags $F\ell(1, 2, \ldots, n-1, V)$ (or more generally for G/B, where G is an arbitrary connected reductive algebraic group), and then it allows a non-difficult generalization for any flag variety $F\ell(i_1, \ldots, i_s, V)$ (more generally to G/P). This more general result yields (10.3) as a very special case.

As a next step, let's discuss the line bundles on $F\ell(1, 2, \ldots, n-1, V)$. It is easy to see that any such line bundle $\mathcal{L}$ is isomorphic to a tensor product of the form $S_1^{k_1} \otimes (S_2/S_1)^{k_2} \otimes \cdots \otimes (S_{n-1}/S_{n-2})^{k_{n-1}}$ for $(k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}$, where S_j is the tautological bundle of rank j on $F\ell(1, 2, \ldots, n-1, V)$; the fiber of S_j at the point $V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V$ of $F\ell(1, 2, \ldots, n-1, V)$ is the space V_j.

Moreover, for every $1 \leq m \leq n-1$, any line bundle on $F\ell(1, 2, \ldots, m, V)$ is isomorphic to $S_1^{k_1} \otimes (S_2/S_1)^{k_2} \otimes \cdots \otimes (S_m/S_{m-1})^{k_m}$ for some $(k_1, \ldots, k_m) \in \mathbb{Z}^m$. This can be proved by induction on m, using a standard fact in algebraic geometry. Indeed, if $m = 1$, $F\ell(1, V) = \mathbb{P}(V)$ and the statement is already known. Assume that the statement holds for $m-1$, and let $\mathcal{L}$ be a line bundle on $F\ell(1, 2, \ldots, m, V)$. Consider the fibration

$$F\ell(1, 2, \ldots, m, V) \to F\ell(1, 2, \ldots, m-1, V) \tag{10.4}$$

with fiber $\mathbb{P}^{(\dim V)-m}$, and restrict $\mathcal{L}$ to each fiber. On each fiber, the restriction of $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{(\dim V)-m}}(s)$ for some s, and as it turns out, s does not depend on the choice of fiber. This follows for instance from the semicontinuity theorem [Har, Ch.III, Theorem 12.8 (see also Exercise 9)]. Therefore, $\mathcal{L} \otimes (S_m/S_{m-1})^{-s}$ is a line bundle on $F\ell(1, 2, \ldots, m, V)$ that is trivial on each fiber of the fibration (10.4), and is hence the pull-back of a line bundle on $F\ell(1, 2, \ldots, m-1, V)$. Since the latter is isomorphic to $S_1^{k_1} \otimes (S_2/S_1)^{k_2} \otimes \cdots \otimes (S_{m-1}/S_{m-2})^{k_{m-1}}$ by the induction assumption, $\mathcal{L}$ itself is isomorphic to $(S_1^{k_1}) \otimes (S_2/S_1)^{k_2} \otimes \cdots \otimes (S_m/S_{m-1})^{k_m}$ for $k_m = -s$.

For a line bundle $\mathcal{L}$ on $F\ell(1, \ldots, m, V)$, we denote by $\mathcal{L}_g$ the restriction of the pull-back of $\mathcal{L}$ on $GL(V) \times F\ell(1, 2, \ldots, m, V)$ to the subvariety

$$g \times F\ell(1, 2, \ldots, m, V),$$

via the group action projection

$$GL(V) \times F\ell(1, 2, \ldots, m, V) \to F\ell(1, 2, \ldots, m, V).$$

A line bundle $\mathcal{L}$ is called $GL(V)$-*linearized*, if for every $g, g', g'' \in GL(V)$, there are isomorphisms

$$i_{g(g')^{-1}} : \mathcal{L}_g \to \mathcal{L}_{g'}$$

such that $i_{g(g')^{-1}} i_{g'(g'')^{-1}} = i_{g(g'')^{-1}}$ and $i_1 = Id$.

Now we note that any line bundle $\mathcal{L}$ on $F\ell(1, 2, \ldots, m, V)$ admits a $GL(V)$-linearization. This is because the bundles S_i/S_{i-1}, and the tensor products of the form $S_1^{k_1} \otimes \cdots \otimes (S_m/S_{m-1})^{k_m}$ are equipped with an obvious $GL(V)$-linearization. However, such a $GL(V)$-linearization is not unique. Indeed, let $\mathcal{L} = \mathcal{O}$ be the trivial line bundle on $F\ell(1, 2, \ldots, m, V)$. Then $\mathcal{L}$ is isomorphic to the bundle

$$(S_1)^k \otimes (S_1/S_1)^k \otimes \cdots \otimes (S_m/S_{m-1})^k \otimes (\Lambda^{n-m}(\tilde{V}/S_m))^k,$$

where from now on $\tilde{V}$ stands for the G-linearized bundle $V \otimes \mathcal{O}$ and k is an arbitrary integer. For each k, the latter bundle has a distinct $GL(V)$-linearization, and hence $\mathcal{O}$ has $GL(V)$-linearizations parameterized by an arbitrary integer. It is not difficult to see that these are all $GL(V)$-linearizations of $\mathcal{O}$ (prove it!) and that moreover, any line bundle on $F\ell(1, 2, \ldots, m, V)$ admits exactly as many G-linearizations as $\mathcal{O}$. This discussion is an informal preparation for the following proposition.

Proposition 10.2 *For any connected reductive group G and any connected subgroup $P \subset G$, the G-linearized line bundles on G/P are in a natural bijection with the group* $\mathrm{Hom}(P, \mathbb{C}^\times)$ *of characters of P, where $\mathbb{C}^\times$ stands for the multiplicative group of $\mathbb{C}$.*

We will not give a formal proof of the proposition, but we will only indicate how one obtains the bijection. Given a G-linearized line bundle, its fiber over the point P is a P-module. Since this P-module is one-dimensional, it is in fact a character of P. Conversely, given a character λ of P and the corresponding one-dimensional P-module $\mathbb{C}_\lambda$, the respective bundle is constructed as the fibered product $G \times_P \mathbb{C}_\lambda$ and has a natural projection onto G/P. The reader should try now to reconstruct the proof of Proposition 10.2 or to read it in the literature.

More generally, one can consider G-linearized vector bundles of arbitrary length, and these of course form a natural category. We leave it to the reader to define morphisms of G-linearized bundles and moreover to prove that the category of G-linearized vector bundles on G/P is equivalent to the category of finite-dimensional P-modules. This equivalence extends the bijection from Proposition 10.2.

Back in the case when $G = GL(V)$ and P is the stabilizer of a flag of subspaces of V of respective dimensions $1, 2, \ldots, m$, the line bundle

$$S_1 \otimes (S_2/S_1) \otimes \cdots \otimes S_m/S_{m-1} \otimes \Lambda^{n-m}(\tilde{V}/S_m) \tag{10.5}$$

corresponds to the character $\det : P \to \mathbb{C}^\times$ which extends to a character of G (check this!). That's why (10.5) is the pull-back on G/P of a G-linearized bundle on G/G, and is hence trivial as a line bundle. The corresponding G-linearized bundle on G/G is not trivial as a G-bundle (since $\det : G \to \mathbb{C}^\times$ does not equal the trivial homomorphism), and (10.5) is also not trivial as a G-bundle on G/P.

Now consider in more detail the case when $m = n - 1$ and $P = B$ is the Borel subgroup of upper-triangular matrices in $GL(V)$ for $V = \mathbb{C}^n$. Here the correspondence from Proposition 10.2 has the following explicit form. Order the basis of $\mathbb{C}^n$ as $e_1, \dots, e_n$. Then the group of characters $\mathrm{Hom}(B, \mathbb{C}^\times)$ is identified with $\mathbb{Z}^n$. The G-linearized line bundle corresponding to $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ is simply

$$S_1^{\lambda_1} \otimes (S_2/S_1)^{\lambda_2} \otimes \cdots \otimes (\tilde{\mathbb{C}}^n/S_{n-1})^{\lambda_n}.$$

Verifying this is a non-difficult but essential computation. The simplest case is $n = 2$: here one needs to check that the B-character in the fiber of the bundle $S_1^{\lambda_1} \otimes (\tilde{\mathbb{C}}^2/S_1)^{\lambda_2}$ at the point $B \in G/B$ is precisely the character $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$. We hope that the reader will verify this explicitly.

For a general connected reductive group G and a Borel subgroup $B \subset G$, we denote by $\mathcal{O}(\lambda)$ the G-linearized line bundle on G/B corresponding to a character $\lambda : B \to \mathbb{C}^\times$, i.e., such that B acts via λ in the fiber of $\mathcal{O}(\lambda)$ at the point $B \in G/B$. If we choose G to be $SL(V)$, then every line bundle on G/B admits a unique G-linearization and the bijection of Proposition 10.2 induces an equality $\mathrm{Pic}\, G/B = \mathrm{Hom}(B, \mathbb{C}^\times)$, where B is now a Borel subgroup of $SL(V)$.

The isomorphisms (10.3) enable us to compute explicitly the cohomology of any $GL(2)$-linearized line bundle on $GL(2)/P = \mathbb{P}(V)$ for $V = \mathbb{C}^2$. Indeed, notice that in this case

$$\mathcal{O}(\lambda) = S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2} = S_1^{\lambda_1 - \lambda_2} \otimes (\Lambda^2(\tilde{V}))^{\otimes \lambda_2},$$

where $(\Lambda^2(\tilde{V}))^{\otimes \lambda_2}$ is a trivial bundle on $\mathbb{P}(V)$ with a nontrivial G-linearization. Hence (10.3) implies for $\lambda_2 - \lambda_1 \geq 0$,

$$\begin{aligned} H^0(\mathbb{P}(V), S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2}) &= S^{\lambda_2 - \lambda_1}(V^*) \otimes (\Lambda^2(V))^{\otimes \lambda_2}, \\ H^1(\mathbb{P}(V), S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2}) &= 0; \end{aligned} \tag{10.6}$$

for $\lambda_2 - \lambda_1 \leq -2$,

$$\begin{aligned} H^0(\mathbb{P}(V), S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2}) &= 0, \\ H^1(\mathbb{P}(V), S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2}) &= S^{\lambda_1 - \lambda_2 - 2}(V) \otimes (\Lambda^2(V))^{\otimes \lambda_2 + 1}; \end{aligned} \tag{10.7}$$

and for $\lambda_2 - \lambda_1 = -1$,

$$H^0(\mathbb{P}(V), S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2}) = 0, \qquad (10.8)$$
$$H^1(\mathbb{P}(V), S_1^{\lambda_1} \otimes (\tilde{V}/S_1)^{\lambda_2}) = 0.$$

10.1.2 The Main Result

Let $\mathfrak{g}$ be a reductive Lie algebra, let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra, and let μ be a weight, i.e., $\mu \in \mathfrak{h}^*$ for some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. We say that μ is $\mathfrak{b}$-*dominant* if $\frac{2\mathrm{Re}(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{R}_{\geq 0}$ where $(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$. By definition a weight μ is $\mathfrak{b}$-*antidominant* if $-\mu$ is $\mathfrak{b}$-dominant. When we consider a connected reductive algebraic group G with Lie algebra $\mathfrak{g}$, we call a character μ of a Borel subgroup $B \subset G$, B-*dominant* or B-*antidominant* if μ, considered as a weight, is $\mathfrak{b}$-dominant or $\mathfrak{b}$-antidominant where $\mathfrak{b} = \mathrm{Lie}\, B$. By $L_B(\mu)$, we denote the simple finite-dimensional G-module with B-highest weight μ. This module is well defined for a B-dominant character μ of B.

We are now ready to state the general Bott–Borel–Weil Theorem.

Theorem 10.3 *Let G be a connected reductive algebraic group, let $B \subset G$ be a Borel subgroup, and let $\lambda : B \to \mathbb{C}^\times$ be a character of B. Let ρ denote the half-sum of roots of $\mathfrak{b} = \mathrm{Lie}\, B$, and let $w(-\lambda + \rho)$ be a $\mathfrak{b}$-dominant weight for some Weyl group element $w \in W$, where W is the Weyl group of the Lie algebra $\mathfrak{g}$ of G. Then if w is not unique*

$$H^{\cdot}(G/B, \mathcal{O}(\lambda)) = 0,$$

while if w is unique

$$H^k(G/B, \mathcal{O}(\lambda)) = \begin{cases} L_B(w(-\lambda + \rho) - \rho)^* \text{ for } k = \ell(w) \\ 0 \quad \text{for } k \neq \ell(w) \end{cases},$$

where $\ell(w)$ denotes the length of w with respective to the simple roots of $\mathfrak{b}$.

In what follows we present a sketch of Demazure's proof of this theorem. For explicitness, we assume that $G = GL(n)$. Then $\lambda = \sum_{i=1}^n \lambda_i$ for some $\lambda_i \in \mathbb{Z}$, and W is the symmetric group on n letters. The reader should then read Demazure's paper [De1] and realize that everything we are saying about $GL(n)$ can in fact be easily said more abstractly for any connected reductive algebraic group G. We will comment on this after we finish the proof.

Sketch of Proof of Theorem 10.3 Let B be the group of upper triangular matrices, set $X = G/B$ and let $p_i : X \to X_i$ denote the projection

$$F\ell(1, 2, \ldots, n-1, \mathbb{C}^n) \to F\ell(1, 2, \ldots i-1, i+1, \ldots, n-1, \mathbb{C}^n).$$

Note that p_i realizes X as a relative projective line X_i. We will compute $H^{\cdot}(X, \mathcal{O}(\lambda))$ by using the Leray spectral sequence of each projection p_i. For an introduction to spectral sequences the reader can consult [GH] or [Wei]. In the case of $\mathcal{O}(\lambda)$ and p_i, the Leray spectral sequence is a spectral sequence with $E_2^{p,q}$-term $H^p(X_i, R^q(p_i)_*\mathcal{O}(\lambda))$ and it converges to $H^{\cdot}(X, \mathcal{O}(\lambda))$. Here $(p_i)_*$ stands for the direct image of $\mathcal{O}$-modules along p_i and $R^q(p_i)_*$ denotes the qth right derived functor of $(p_i)_*$. Since the dimension of the fiber of p_i equals 1, $R^q(p_i)_*\mathcal{O}(\lambda) = 0$ for $q > 1$, so there can be no more than two nonvanishing direct image sheaves. In fact, a crucial observation is that for each pair (λ, i), there is at most one nonvanishing direct image sheaf among $R^0(p_i)_*\mathcal{O}(\lambda)$ and $R^1(p_i)_*\mathcal{O}(\lambda)$. Indeed,

$$\begin{aligned}\mathcal{O}(\lambda) &= S_1^{\lambda_1} \otimes (S_2/S_1)^{\lambda_2} \otimes \cdots \otimes (\tilde{\mathbb{C}}^n/S_{n-1})^{\lambda_n} \\ &= p_i^*(S_1^{\lambda_1} \otimes \cdots \otimes (S_{i-1}/S_{i-2})^{\lambda_{i-1}} \otimes (S_{i+2}/S_{i+1})^{\lambda_{i+2}} \otimes \cdots \otimes \\ &\qquad (\tilde{\mathbb{C}}^n/S_{n-1})^{\lambda_n}) \otimes (S_i/S_{i-1})^{\lambda_i} \otimes (S_{i+1}/S_i)^{\lambda_{i+1}}.\end{aligned}$$

Moreover, the bundles S_i/S_{i-1} and S_{i+1}/S_i are the relative analogs (along the relative projective line p_i) of the line bundles S_1 and $\tilde{\mathbb{C}}^2/S_1$ on $\mathbb{P}^1$, and using some standard tools of algebraic geometry, it is not difficult to show that relative versions of the isomorphisms (10.6), (10.7), (10.8) hold. More precisely, we have for $\lambda_{i+1} - \lambda_i \geq 0$,

$$\begin{aligned}R^0(p_i)_*\mathcal{O}(\lambda) &= S^{\lambda_{i+1}-\lambda_i}((S_{i+1}/S_{i-1})^*) \otimes \left(\Lambda^2(S_{i+1}/S_{i-1})\right)^{\lambda_{i+1}} \\ &\quad \otimes (S_1^{\lambda_1} \otimes \cdots \otimes (S_{i-1}/S_{i-2})^{\lambda_{i-1}} \otimes \\ &\quad (S_{i+2}/S_{i+1})^{\lambda_{i+2}} \otimes \cdots \otimes (\tilde{\mathbb{C}}^n/S_{n-1})^{\lambda_n}), \\ R^1(p_i)_*\mathcal{O}(\lambda) &= 0;\end{aligned}$$

for $\lambda_{i+1} - \lambda_i \leq -2$,

$$\begin{aligned}R^0(p_i)_*\mathcal{O}(\lambda) &= 0, \\ R^1(p_i)_*\mathcal{O}(\lambda) &= S^{\lambda_i-\lambda_{i+1}-2}(S_{i+1}/S_{i-1}) \otimes \left(\Lambda^2(S_{i+1}/S_{i-1})\right)^{\lambda_{i+1}+1} \\ &\quad \otimes (S_1^{\lambda_1} \otimes \cdots \otimes (S_{i-1}/S_{i-2})^{\lambda_{i-1}} \otimes \\ &\quad (S_{i+2}/S_{i+1})^{\lambda_{i+2}} \otimes \cdots \otimes (\tilde{\mathbb{C}}^n/S_{n-1})^{\lambda_n});\end{aligned}$$

for $\lambda_{i+1} - \lambda_i = -1$,

$$R^0(p_i)_* \mathcal{O}(\lambda) = R^1(p_i)_* \mathcal{O}(\lambda) = 0.$$

This implies immediately that the Leray spectral sequence of $\mathcal{O}(\lambda)$ degenerates and that

$$H^k(X, \mathcal{O}(\lambda)) = H^k(X_i, R^0(p_i)_* \mathcal{O}(\lambda)) \text{ for } \lambda_{i+1} - \lambda_i \geq 0,$$

$$H^k(X, \mathcal{O}(\lambda)) = H^{k-1}(X_i, R^1(p_i)_* \mathcal{O}(\lambda)) \text{ for } \lambda_{i+1} - \lambda_i \leq -2,$$

and

$$H^k(X, \mathcal{O}(\lambda)) = 0 \text{ for all } k \geq 0 \text{ when } \lambda_{i+1} - \lambda_i = -1. \tag{10.9}$$

Consider the case when $\lambda_{i+1} - \lambda_i \geq 0$. Let w_i denote the permutation that interchanges the ith and $(i+1)$th coordinate of a weight. Then $w_i(\lambda - \rho) + \rho = \lambda_i \ldots \lambda_{i-1}, \lambda_{i+1} + 1, \lambda_i - 1, \lambda_{i+2}, \ldots, \lambda_n$, and we claim that

$$R^0(p_i)_* \mathcal{O}(\lambda) = R^1(p_i)_* \mathcal{O}(w_i(\lambda - \rho) + \rho). \tag{10.10}$$

Indeed, the two visually different factors of the left- and right-hand side of (10.10) are respectively

$$S^{\lambda_{i+1} - \lambda_i}((S_{i+1}/S_{i-1})^*) \otimes \left(\Lambda^2(S_{i+1}/S_{i-1})\right)^{\lambda_{i+1}}$$

and

$$S^{\lambda_{i+1} - \lambda_i}(S_{i+1}/S_{i-1}) \otimes \left(\Lambda^2(S_{i+1}/S_{i-1})\right)^{\lambda_i}.$$

However these G-linearized bundles are canonically isomorphic as

$$(S_{i+1}/S_{i-1})^* = (S_{i+1}/S_{i-1}) \otimes \Lambda^2((S_{i+1}/S_{i-1})^*).$$

The isomorphism (10.10) now allows us to conclude that for $\lambda_{i+1} - \lambda_i \geq 0$,

$$H^k(X, \mathcal{O}(\lambda)) = H^{k+1}(X, \mathcal{O}(w_i(\lambda - \rho) + \rho)) \text{ for all } k \geq 0. \tag{10.11}$$

Since the Borel subgroup B is fixed throughout the proof, in what follows we refer to weights just as dominant or antidominant.

Now let λ be antidominant, i.e., $\lambda_{i+1} - \lambda_i \geq 0$ for $1 \leq i \leq n$. Consider the permutation of maximal length $w_{\max}$ that sends $\{1, 2, \ldots, n\}$ to $\{n, n-1, \ldots, 1\}$, and decompose it as $w_{n-1} \circ w_{n-2} \circ w_{n-1} \circ \cdots \circ w_2 \circ \cdots \circ w_{n-1} \circ w_1 \circ w_2 \circ \cdots \circ w_{n-1}$. By a truncation of $w_{\max}$, we mean the application of consecutive reflections

w_i starting with the rightmost one up to some w_k. Then for each truncation w of $w_{\max}$ of the form $w = w_k \circ \cdots \circ w_{n-1}$ for some $1 \leq k \leq n-1$, it is true that $(w_k \circ w(\lambda - \rho) + \rho)_{k+1} - (w_k \circ w(\lambda - \rho) + \rho)_k \geq 0$. (Check this!) Therefore we can iterate the isomorphism (10.11), and this yields

$$H^k(X, \mathcal{O}(\lambda)) = H^{k+\ell(w_{\max})}(X, \mathcal{O}(w_{\max}(\lambda - \rho) + \rho)), \tag{10.12}$$

where $l(w_{\max})$ is the length of $w_{\max}$ which equals $\frac{n(n-1)}{2} = \dim X$. The isomorphism (10.12) implies immediately that

$$H^k(X, \mathcal{O}(\lambda)) = 0 \text{ for } k \geq 1$$

whenever λ is antidominant.

If λ is not antidominant, then there is always an antidominant weight μ in the set $\{w(\lambda - \rho) \mid w \in W\}$. If μ is regular, i.e., w is unique with the property that $w(\lambda - \rho)$ is antidominant, we notice that $\lambda' := w(\lambda - \rho) + \rho$ is also antidominant. By decomposing w^{-1} into a product of simple transpositions w_i and iterating (10.11), we obtain

$$H^0(X, \mathcal{O}(\lambda')) = H^{\ell(w^{-1})}(X, \mathcal{O}(\lambda)),$$
$$H^k(X, \mathcal{O}(\lambda)) = 0 \text{ for } k \neq \ell(w^{-1}).$$

Since $w(\lambda - \rho)$ is antidominant if and only if $w(-\lambda + \rho)$ is dominant, we have now shown that if w is the unique permutation for which $w(-\lambda + \rho) - \rho$ is dominant, then

$$H^k(X, \mathcal{O}(\lambda)) = \begin{cases} 0 \text{ for } k \neq l(w) \\ H^0(X, \mathcal{O}(w(\lambda - \rho) + \rho)) \text{ for } k = l(w). \end{cases} \tag{10.13}$$

The antidominant representative in the set $\{w(\lambda - \rho) \mid w \in W\}$ is not regular if and only if $\lambda_{i+j} - \lambda_i = -j$ for some i and j. Choose j minimal for a given i. Then we have

$$(w'(\lambda - \rho) + \rho)_{i+1} - (w'(\lambda - \rho) + \rho)_i = -1$$

where $w' = w_{i+j-1} \circ \cdots \circ w_{i+1}$. By using the fact

$$(w'(\lambda - \rho) + \rho)_{i+t} - (w'(\lambda - \rho) + \rho)_i \neq -t$$

for $2 \leq t \leq i + j$, we see, via an iteration of the isomorphism (10.11), that

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k+s}(G/B, \mathcal{O}(w'(\lambda - \rho) + \rho))$$

for some s. However, we already know by (10.9) that

$$H^{\cdot}(G/B, \mathcal{O}(w'(\lambda - \rho) + \rho)) = 0,$$

hence also

$$H^{\cdot}(G/B, \mathcal{O}(\lambda)) = 0.$$

It remains to show that $H^0(X, \mathcal{O}(\lambda)) = L_B(-\lambda)^*$ for an antidominant λ. Note that if $H^0(X, \mathcal{O}(\lambda)) \neq 0$, the canonical isomorphism of evaluating a global section at a closed point induces a morphism of G-linearized bundles

$$H^0(X, \mathcal{O}(\lambda)) \otimes \mathcal{O}_X \to \mathcal{O}(\lambda). \tag{10.14}$$

This morphism induces a morphism of B-modules

$$H^0(X, \mathcal{O}(\lambda)) \to \mathbb{C}_\lambda, \tag{10.15}$$

whose kernel contains a G-submodule L', unless $H^0(X, \mathcal{O}(\lambda)) = L_B(-\lambda)^*$ and the morphism (10.13) maps $L_B(-\lambda)^*$ to its B-lowest weight λ. (Please prove this!) However the existence of a simple G-constituent L' of $H^0(X, \mathcal{O}(\lambda))$ which maps to zero via (10.13) is contradictory, as the global sections of $\mathcal{O}(\lambda)$ lying in L' would have zero values at all closed points of X, and hence would be zero. Therefore $H^0(X, \mathcal{O}(\lambda)) \neq 0$ implies $H^0(X, \mathcal{O}(\lambda)) = L_B(-\lambda)^*$. □

The isomorphism (10.11) is probably the most important ingredient of the above proof. In the case of a general reductive algebraic group, it is replaced by the isomorphism

$$H^k(G/B, \mathcal{O}(\lambda) = H^{k+1}(G/B, \mathcal{O}(w_\alpha(\lambda - \rho) + \rho)) \tag{10.16}$$

for any simple root α of B such that $\frac{2(\lambda-\rho,\alpha)}{(\alpha,\alpha)} \leq 0$ and any $k \in \mathbb{Z}_{\geq 0}$. To prove (10.16), one applies arguments similar to the above, to the $\mathbb{P}^1$-fibration

$$p_\alpha : G/B \to G/P_\alpha$$

where P_α is the parabolic subgroup of G obtained from B via inverting the root α.

We complete this section with an even more general version of the Bott–Borel–Weil Theorem which applies to any simple G-linearized bundle on G/P. Let G be a connected reductive algebraic group and let P be any parabolic subgroup. If E is a G-linearized vector bundle on G/P of rank r, where r is possibly greater than 1, we call E simple if E has no proper G-linearized subbundle. The correspondence of Proposition 10.2 can be extended to arbitrary finite-dimensional P-modules. We state this fact as a proposition whose proof we leave to the reader.

Proposition 10.4 *For any connected algebraic group G and any parabolic subgroup $P \subset G$, the isomorphism classes of G-linearized vector bundles E on G/P are in bijection with the isomorphism classes of finite-dimensional P-modules. Simple G-linearized bundles correspond to simple P-modules.*

Given a simple finite-dimensional P-module, let $\mathcal{O}_{G/P}(E)$ denote the respective G-linearized bundle on G/P. The following theorem reduces the computation of $H^{\cdot}(G/P, \mathcal{O}_{G/P}(E))$ to Theorem 10.3.

Theorem 10.5 *Let G be a connected reductive algebraic group, let P be a parabolic subgroup, and let $B \subset P$ be a Borel subgroup. Let E be a simple finite-dimensional P-module with B-lowest weight λ (i.e., the 1-dimensional B-module $\mathbb{C}_\lambda$ is a quotient of E considered as a B-module). Then*

$$H^{\cdot}(G/P, \mathcal{O}_{G/P}(E)) = H^{\cdot}(G/B, \mathcal{O}(\lambda)).$$

Proof The key idea is that

$$\mathcal{O}_{G/P}(E) = p_*\mathcal{O}(\lambda), \tag{10.17}$$

where $p : G/B \to G/P$ is the canonical surjective morphism. Indeed, there is a natural epimorphism of G-linearized vector bundles on G/B

$$\pi : p^*\mathcal{O}_{G/P}(E) \to \mathcal{O}(\lambda), \tag{10.18}$$

which corresponds to the surjection of B-modules $E \to \mathbb{C}_\lambda$ at the point $B \in G/B$. This yields a morphism of sheaves of $\mathcal{O}_{G/P}$-modules

$$\pi' : \mathcal{O}_{G/P}(E) = p_*p^*\mathcal{O}_{G/P}(E) \to p_*\mathcal{O}(\lambda).$$

By a well-known theorem of Grauert [Har, Ch.III, Corollary 12.9], $p_*\mathcal{O}(\lambda)$ is a vector bundle of rank equal to $\dim E$. Therefore π' is a morphism of vector bundles which is an isomorphism on each fiber, and hence is itself an isomorphism. Thus (10.17) holds as claimed.

The proof is completed by one more straightforward application of the Bott–Borel–Weil theorem and the Leray spectral sequence. Indeed, all we have to check is that

$$H^{\cdot}(G/P, p_*\mathcal{O}(\lambda)) = H^{\cdot}(G/B, \mathcal{O}(\lambda)), \tag{10.19}$$

and it follows from the Leray spectral sequence that a sufficient condition for (10.19) to hold is the vanishing of all higher direct image sheaves $R^i p_*\mathcal{O}(\lambda)$ for $i > 0$. As $R^0 p_*\mathcal{O}(\lambda)$ is nonzero by definition, the vanishing of $R^i p_*\mathcal{O}(\lambda)$ for $i > 0$ follows again from Grauert's Theorem and from the Bott–Borel–Weil Theorem applied to the restriction $\mathcal{O}(\lambda)_{P'/B'}$ to the fibers P'/B' of the surjective morphism p. This completes the proof. □

10.2 Toward a Bott–Borel–Weil Theorem for Lie Supergroups

In this section we present a Bott–Borel–Weil Theorem for finite-dimensional classical Lie supergroups in the special case that the weights under consideration are typical. The condition of typicality is an "odd counterpart" to the condition of regularity for a weight of a reductive algebraic group. The finite-dimensional representations with typical highest weights are precisely the representations that admit a Kac–Weyl character formula similar to the classical Weyl character formula for Lie algebras. Generalizing the Bott–Borel–Weil Theorem to arbitrary weights remains an open problem.

10.2.1 Preliminaries on Lie Supergroups and Flag Supermanifolds

In what follows we work with $\mathbb{Z}_2$-graded vector spaces $L = L_{\bar{0}} \oplus L_{\bar{1}}$ and write their dimensions as $m|n$ where $m = \dim L_{\bar{0}}$, $n = \dim L_{\bar{1}}$. The same convention applies to the rank of $\mathbb{Z}_2$-graded vector bundles $E = E_{\bar{0}} \oplus E_{\bar{1}}$ on algebraic varieties. By Π, we denote the *parity changing functor*: if $L = L_{\bar{0}} \oplus L_{\bar{1}}$, then $(\Pi L)_{\bar{0}} = L_{\bar{1}}$, $(\Pi L)_{\bar{1}} = L_{\bar{0}}$. Clearly, the parity changing functor is defined also on arbitrary sheaves of $\mathbb{Z}_2$-graded abelian groups. A *superalgebra* simply means a $\mathbb{Z}_2$-graded algebra.

A *supermanifold* M of dimension $m|n$ can be defined as a ringed space $(M_{red}, \mathcal{O}_M)$ where M_{red} is a manifold of dimension m and $\mathcal{O}_M$ is a sheaf of superalgebras on M_{red} that is locally isomorphic as a sheaf of superalgebras to the Grassmann algebra of a vector bundle of rank n on M_{red}, see [Be, Ko2, L, Man] and the references therein. In what follows we work in the category of algebraic supermanifolds, or more generally in the category of superschemes as introduced by Yu. Manin in [Man]. For a true understanding of the material of this section, the readers should familiarize themselves with some main results of [Man].

By definition, the manifold M_{red} is the *underlying manifold* or *reduced manifold* of M, and is recovered from M as the ringed space $(M_{red}, \mathcal{O}_{M_{red}} := \mathcal{O}_M/\mathcal{N})$, where $\mathcal{N}$ is the subsheaf of nilpotent local sections in $\mathcal{O}_M$. Clearly, M_{red} is a closed subsupermanifold of M of dimension $m|0$.

A *vector bundle* E on a supermanifold M can be defined as a locally free sheaf of $\mathcal{O}_M$-modules of rank $k|l$, i.e., as a $\mathbb{Z}_2$-graded sheaf locally isomorphic to $\mathcal{O}_M^{\oplus m} \oplus (\Pi\mathcal{O}_M)^{\oplus n}$. The restriction $E_{red} := E/\mathcal{N}E$ of E to M_{red} is a $\mathbb{Z}_2$-graded usual vector bundle on M_{red} of rank $k|l$. If M has dimension $m|n$, then the sheaf $\mathcal{N}/\mathcal{N}^2$ is a vector bundle on M_{red} of rank $0|n$ (odd vector bundle) and is by definition the *conormal bundle* $\mathcal{N}^*_{M_{red},M}$ of M_{red} in M. The structure sheaf $\mathcal{O}_M$ is locally isomorphic to the sheaf of superalgebras $\Lambda^\cdot(\mathcal{N}/\mathcal{N}^2)$, but there need not be a global isomorphism. In the case that a global isomorphism exists, we call

the supermanifold *splitting*, and otherwise *non-splitting*. In the category of C^∞-supermanifolds, which we do not consider here, any supermanifold is splitting.

One of the reasons why algebraic supermanifolds are interesting and quite difficult to study is that they are in general non-splitting. This applies in particular to the flag supermanifolds discussed below.

An *(algebraic) Lie supergroup* G is a group object in the category of supermanifolds, that is G is an (algebraic) supermanifold with a multiplication morphism $\mu : G \times G \to G$, an inversion morphism $i : G \to G$ and a unit morphism $e : 1 \to G$ that make G a group object in the category of supermanifolds.

An equivalent definition of a Lie supergroup is sometimes handier to work with. Namely, one can consider pairs $(G', \mathfrak{g})$ where G' is an ordinary (algebraic) Lie group, $\mathfrak{g}$ is a finite-dimensional Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \operatorname{Lie} G'$ (i.e., $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G') and $\mathfrak{g}$ is endowed with the structure of a G'-module which induces the adjoint $\mathfrak{g}_{\bar{0}}$-module structure on $\mathfrak{g}$.

It is a theorem that the category of (algebraic) Lie supergroups is canonically equivalent to the category of pairs as above [S1, Vi]. In particular, one can define classical Lie supergroups $GL(m|n)$, $SOSp(m|2n)$, $\mathbf{P}(n)$, $Q(n)$, as respective pairs $(GL(m) \times GL(n), \mathfrak{gl}(m|n))$, $(SO(m) \times Sp(2n), \mathfrak{osp}(m|2n))$, $(GL(n), \mathbf{p}(n))$, $(GL(n), \mathfrak{q}(n))$, and in each case the G'-module structure on $\mathfrak{g}$ is obvious from the structure of the roots of $\mathfrak{g}$.

Moreover this approach allows us to easily define Borel and parabolic subsupergroups of the classical Lie supergroups. By definition, a *parabolic subsupergroup* of G is a supergroup of the form $P = (P_{red}, \mathfrak{p})$, where P_{red} is a parabolic subgroup of G_{red} and $\mathfrak{p}$ is a parabolic subsuperalgebra of $\mathfrak{g}$. In particular, a *Borel subsupergroup* $B = (B_{red}, \mathfrak{b})$ is a subsupergroup B of G, where B_{red} is a Borel subgroup of G_{red} and $\mathfrak{b}$ is a Borel subsuperalgebra of $\mathfrak{g}$.

Recall that a Cartan subsuperalgebra $\mathfrak{h}$ of a classical Lie superalgebra $\mathfrak{g}$ can be described as the centralizer of a Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ (see Theorem 2.7). For $\mathfrak{g} \neq \mathfrak{q}(n)$, a Cartan subsuperalgebra of $\mathfrak{g}$ is simply a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. In the rest of this section, we consider only Borel subsuperalgebras $\mathfrak{b} \subset \mathfrak{g}$ containing a fixed Borel subalgebra $\mathfrak{b}_{\bar{0}}$ and a fixed Cartan subsuperalgebra $\mathfrak{h}$ such that $\mathfrak{h}_{\bar{0}} \subset \mathfrak{b}_{\bar{0}}$. Concerning Borel and parabolic subsupergroups, we only consider such subsupergroups of G whose reduced manifold G_{red} contains a fixed maximal torus H_{red} and a fixed Borel subgroup B_{red} with $B_{red} \supset H_{red}$.

Two Borel subsupergroups B_1, B_2 of G are *connected by an odd reflection* (or more generally, *adjacent*) if the respective Borel subsuperalgebras $\mathfrak{b}_1$, $\mathfrak{b}_2$ are connected by an odd reflection (respectively, adjacent). By applying Lemma 2.11, we see that any two Borel subsupergroups that we consider can be connected by a chain of odd reflections (possibly degenerate for $G = \mathbf{P}(n)$).

Recall that for $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, $\mathbf{p}(n)$ every root space $\mathfrak{g}^\alpha$ has dimension $1|0$ or $0|1$, allowing us to call the roots even or respectively odd. For $\mathfrak{osp}(m|2n)$, there are odd roots such that 2α is an even root: $\alpha = \delta_i$ in the notation of Sect. 2.5. For $\mathfrak{g} = \mathfrak{q}(n)$, all root spaces have dimension $1|1$, and we do not refer to the roots

as even or odd. Given a Borel subsuperalgebra $\mathfrak{b} \subset \mathfrak{g}$, we set

$$\rho_{\mathfrak{b}} := \frac{1}{2} \sum_{\text{even roots of } \mathfrak{b}} \alpha - \frac{1}{2} \sum_{\substack{\text{odd roots of } \mathfrak{b} \\ \text{such that } -\alpha \\ \text{is a root of } \mathfrak{g}}} \alpha \quad \text{for } \mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n), \mathbf{p}(n),$$

$$\rho_{\mathfrak{b}} := 0 \quad \text{for } \mathfrak{g} = \mathfrak{q}(n).$$

By definition the *weights* of $\mathfrak{g}$ are elements λ of $\mathfrak{h}_{\bar{0}}^*$, and we can express a weight λ as

$$\lambda = \sum_{i=1}^{k} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j \tag{10.20}$$

in the notation of Sect. 2.5, where $k = m$ for $\mathfrak{gl}(m|n)$ and $k = \lfloor m/2 \rfloor$ for $\mathfrak{osp}(m|2n)$. The coordinates a_i, b_j are the *marks* of λ. Note that the vector $\rho_{\mathfrak{b}}$ is a weight, i.e., $\rho_{\mathfrak{b}} \in \mathfrak{h}_{\bar{0}}^*$.

For $\mathfrak{g} = \mathfrak{q}(n)$, $\dim \mathfrak{h} = n|n$. For $\mathfrak{g} \neq \mathfrak{q}(n)$, any weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$ defines a $\mathfrak{h}$-module $\mathbb{C}_\lambda$ of dimension $1|0$, and $\mathbb{C}_\lambda$ can be made into a $\mathfrak{b} = \mathfrak{h} \dot{+} \mathfrak{n}$-module by putting $\mathfrak{n} \cdot \mathbb{C}_\lambda = 0$. For $\mathfrak{g} = \mathfrak{q}(n)$ and every weight λ, the algebra $U(\mathfrak{h})/(\ker \lambda)$ is a Clifford algebra, and one can check that for each λ, there is an irreducible $\mathfrak{h}$-module $\boldsymbol{\lambda}$ such that $\mathfrak{h}_{\bar{0}}$ acts by λ. The dimension of $\boldsymbol{\lambda}$ is given by

$$\dim \boldsymbol{\lambda} = \begin{cases} 1|0 \text{ or } 0|1 & \text{if } \#\lambda = 0 \\ 2^{d-1}|2^{d-1} & \text{if } \#\lambda = 2d \text{ and } d > 0 \\ 2^d|2^d & \text{if } \#\lambda = 2d+1 \end{cases} \tag{10.21}$$

where $\#\lambda$ is the number of nonzero marks of λ [P1]. The representation $\boldsymbol{\lambda}$ is determined up to parity change, and for our purposes this is sufficient (we leave it to the reader to determine for which weights λ the representation $\boldsymbol{\lambda}$ is unique up to isomorphism).

For any $\mathfrak{b}$ and any weight λ, the $\mathfrak{g}$-module $L_{\mathfrak{b}}(\lambda)$ is defined to be the unique (up to isomorphism) simple $\mathfrak{g}$-module with $\mathfrak{b}$-highest weight λ. We choose to define $L_{\mathfrak{b}}(\lambda)$ up to parity change. This means that for $\mathfrak{g} \neq \mathfrak{q}(n)$, we allow the highest weight space of $L_{\mathfrak{b}}(\lambda)$ to be $1|0$-dimensional or $0|1$-dimensional. For $\mathfrak{g} = \mathfrak{q}(n)$, the $\mathfrak{b}$-highest weight space of $L_{\mathfrak{b}}(\lambda)$ is the $\mathfrak{h}$-module $\boldsymbol{\lambda}$, which we defined up to parity change anyway. If $L_{\mathfrak{b}}(\lambda)$ is finite-dimensional, then $L_{\mathfrak{b}}(\lambda)$ is a G-module, i.e., a $(G_{red}, \mathfrak{g})$-module, and in this case we also write $L_B(\lambda)$. As we will now see for the $\mathfrak{g}$-module $L_{\mathfrak{b}}(\lambda)$ to be finite-dimensional, it is in general not sufficient for the simple $\mathfrak{g}_{\bar{0}}$-module $L_{\mathfrak{b}_{\bar{0}}}(\lambda)$ with $\mathfrak{b}_{\bar{0}}$ highest weight λ to be finite dimensional.

Assume that the weight (10.20) is a $\mathfrak{g}_{\bar{0}}$-integral weight, i.e., that $\lambda \in \mathfrak{h}_{\bar{0}}^*$ arises from a homomorphism of algebraic groups $B_{red} \to \mathbb{C}^*$. In terms of the marks a_i, b_j, this integrality condition is equivalent to the requirement that all a_i, b_j be integers. Now let $\mathfrak{b}$ be the standard Borel subsuperalgebra from Definition 2.12. For any integral weight λ, we set $\#\lambda_a := \#\{i \mid a_i \neq 0\}$. Here are the necessary and sufficient conditions on λ for the module $L_{\mathfrak{b}}(\lambda)$ to be finite dimensional:

$$\mathfrak{gl}(m|n), \qquad a_1 \geq \cdots \geq a_m,\ b_1 \geq \cdots \geq b_n \tag{10.22}$$

$$\mathfrak{osp}(2k+1|2n), \qquad a_1 \geq \cdots \geq a_k \geq 0,\ b_1 \geq \cdots \geq b_n \geq \#\lambda_a \tag{10.23}$$

$$\mathfrak{osp}(2|2n), \qquad b_1 \geq \cdots \geq b_n \geq 0 \tag{10.24}$$

$$\mathfrak{osp}(2k|2n),\ k \geq 1, \qquad a_1 \geq \cdots \geq a_{k-1} \geq |a_k|,\ b_1 \geq \cdots \geq b_n \geq \#\lambda_a \tag{10.25}$$

$$\mathbf{p}(n), \qquad a_1 \geq \cdots \geq a_n \tag{10.26}$$

$$\mathfrak{q}(n), \qquad a_1 \geq \cdots \geq a_n, \text{ and if } a_i = a_j, \text{ then } a_i = a_j = 0. \tag{10.27}$$

These conditions should be credited to Kac [K2] for $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n)$, and to the first author [P2] for $\mathfrak{g} = \mathfrak{q}(n)$. For the cases, $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n), \mathbf{p}(n)$ the fact that the conditions (10.22), (10.23), (10.24) are necessary and sufficient is rather obvious. Indeed, note that here the conditions (10.22), (10.23), (10.24) are precisely the conditions for finite-dimensionality of the module $L_{\mathfrak{b}_{\bar{0}}}(\lambda)$. Moreover, in these cases there is a natural $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1$ such that $\mathfrak{b}_{\bar{1}} = \mathfrak{g}_1$. Therefore the induced module $U(\mathfrak{g}) \otimes_{\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1} L_{\mathfrak{b}_{\bar{0}}}(\lambda)$ where $\mathfrak{b}_{\bar{1}} \cdot L_{\mathfrak{b}_{\bar{0}}}(\lambda) = 0$, called a *Kac module*, is finite dimensional if and only if $\dim L_{\mathfrak{b}_{\bar{0}}}(\lambda) < \infty$. This shows that $\dim L_{\mathfrak{b}}(\lambda) < \infty$ if and only if $\dim L_{\mathfrak{b}_{\bar{0}}}(\lambda) < \infty$, since $L_{\mathfrak{b}}(\lambda)$ is isomorphic to a quotient of the Kac module. In the other cases, i.e., $\mathfrak{g} = \mathfrak{osp}(m|2n)$ for $m \neq 2$, $\mathfrak{g} = \mathfrak{q}(n)$, the respective conditions (10.23), (10.25), (10.27) are stronger than the condition that $L_{\mathfrak{b}_{\bar{0}}}(\lambda)$ be finite dimensional.

Next, recall from Sect. 2.2.2 that there is an even nondegenerate supersymmetric invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ of $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n)$. In these cases, a weight μ is *typical* if $(\mu, \alpha) \neq 0$ for any odd root α of $\mathfrak{g}$ such that 2α is not a root. Explicitly, for $\mu = \sum_{i=1}^k c_i \varepsilon_i + \sum_{j=1}^n d_j \delta_j$, this condition for $\mathfrak{g} = \mathfrak{gl}(m|n)$ is $c_i \neq -d_j$ for any i, j, while for $\mathfrak{g} = \mathfrak{osp}(m|2n)$, it is $c_i \neq \pm d_j$ for any i, j. For $\mathfrak{g} = \mathfrak{q}(n)$, a weight $\mu = \sum_{i=1}^n c_i \varepsilon_i$ is *typical* if $c_i \neq -c_j$ for any $i \neq j$. (Equivalently, one can define typicality of a weight for $\mathfrak{g} = \mathfrak{q}(n)$ by using an odd bilinear form on $\mathfrak{h}_{\bar{0}}^*$, see [P2].)

Let W denote the Weyl group of $\mathfrak{g}_{\bar{0}}$. We call a weight μ *regular* (or $\mathfrak{g}$-*regular*) if it is typical and, in addition, no element of W fixes μ. Explicitly, for $\mu = \sum_{i=1}^k c_i \varepsilon_i + \sum_{j=1}^n d_j \delta_j$, this condition for $\mathfrak{g} = \mathfrak{gl}(m|n)$ is $c_i \neq c_j$, $c_i \neq -d_j$, $d_i \neq d_j$, for any i, j, while for $\mathfrak{g} = \mathfrak{osp}(m|2n)$, it is $c_i \neq \pm c_j$, $c_i \neq \pm d_j$, $d_i \neq \pm d_j$ for any i, j. Explicitly, for $\mathfrak{g} = \mathfrak{q}(n)$ and $\mu = \sum_{i=1}^n c_i \varepsilon_i$, this condition is $c_i \neq \pm c_j$ for any i, j.

For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n)$, we call a weight μ $\mathfrak{b}_{\bar{0}}$-*dominant* (or B_{red}-*dominant*) if $\frac{2\,\mathrm{Re}(\mu,\alpha)}{(\alpha,\alpha)} \geq 0$ for all roots of $\mathfrak{b}_{\bar{0}}$. For $\mathfrak{g} = \mathfrak{q}(n)$, a weight μ is

$\mathfrak{b}_{\bar{0}}$-*dominant* if $\frac{2\,\mathrm{Re}(\mu,\alpha)_{\bar{0}}}{(\alpha,\alpha)_{\bar{0}}} \geq 0$ for all roots of $\mathfrak{b}_{\bar{0}}$, where $(\cdot,\cdot)_{\bar{0}}$ is the form on $\mathfrak{h}^*$ induced by the bilinear form $\mathrm{tr}(AB)$ on $\mathfrak{q}(n)_{\bar{0}} = \mathfrak{gl}(n)$. A weight μ is $\mathfrak{b}_{\bar{0}}$-*antidominant* (or B_{red}-*antidominant*) if the weight $-\mu$ is $\mathfrak{b}_{\bar{0}}$-dominant.

The following is a technical lemma which is convenient to have in the proof of Theorem 10.7 below.

Lemma 10.6 *Let* $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n), \mathfrak{q}(n)$. *If* λ *is an integral weight and* $\lambda + \rho_{\mathfrak{b}}$ *is* $\mathfrak{b}_{\bar{0}}$*-dominant and regular, then* $\dim L_{\mathfrak{b}}(\lambda) < \infty$.

Recall that $\mathfrak{b}_{\bar{0}}$ is by assumption the standard Borel subalgebra of $\mathfrak{g}_{\bar{0}}$. Nevertheless, it is clear that Lemma 10.6 is valid for any Borel subsuperalgebra of $\mathfrak{g}$ without this assumption. Indeed, all Cartan subalgebras of $\mathfrak{g}_{\bar{0}}$ are conjugate, and the claim of the lemma is compatible with the action of the Weyl group on the set of Borel subalgebras.

Proof of Lemma 10.6 If $\mathfrak{b}$ is not the standard Borel subsuperalgebra, then we connect $\mathfrak{b}$ to the standard Borel subalgebra $\mathfrak{b}'$ by a chain of odd reflections (see Lemma 2.11). It is a well-known fact that if L is a finite-dimensional simple $\mathfrak{g}$-module with $\mathfrak{b}_1$ highest weight λ_1 and $\lambda_1 + \rho_1$ is typical, then the $\mathfrak{b}_2$-highest weight λ_2 of L equals $\lambda_1 + \rho_{\mathfrak{b}_1} - \rho_{\mathfrak{b}_2}$, and hence $\lambda_1 + \rho_{\mathfrak{b}_1} = \lambda_2 + \rho_{\mathfrak{b}_2}$. This fact is a direct consequence of the Kac–Weyl character formula for the typical finite-dimensional $\mathfrak{g}$-modules [K3]. Therefore in the rest of the proof we assume $\mathfrak{b} = \mathfrak{b}'$ is the standard Borel subsuperalgebra.

We will handle this case by explicitly deriving the necessary and sufficient conditions (given above) on λ for the module $L_{\mathfrak{b}}(\lambda)$ to be finite dimensional from the regularity and $\mathfrak{b}_{\bar{0}}$-dominance of the weight $\lambda + \rho_{\mathfrak{b}}$. For $\mathfrak{q}(n)$ the statement is obvious, since in this case the weight λ is assumed to be $\mathfrak{b}_{\bar{0}}$-dominant. Hence, in the remainder of the proof, we suppose that $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n)$.

The vector $\rho_{\mathfrak{b}}$ for the standard Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is given as follows:

$$\mathfrak{gl}(m|n) \qquad \rho_{\mathfrak{b}} = \sum_{i=1}^{m} \left(\frac{1}{2}(m-n+1) - i \right) \varepsilon_i + \sum_{j=1}^{n} \left(\frac{1}{2}(m+n+1) - j \right) \delta_j$$

$$\mathfrak{osp}(2k+1|2n) \qquad \rho_{\mathfrak{b}} = \sum_{i=1}^{k} \left(k - i + \frac{1}{2} \right) \varepsilon_i + \sum_{j=1}^{n} \left(n + \frac{1}{2} - k - j \right) \delta_j$$

$$\mathfrak{osp}(2|2n) \qquad \rho_{\mathfrak{b}} = -n\varepsilon_1 + \sum_{j=1}^{n} (n+1-j)\, \delta_j$$

$$\mathfrak{osp}(2k|2n) \qquad \rho_{\mathfrak{b}} = \sum_{i=1}^{k} (k-i)\, \varepsilon_i + \sum_{j=1}^{n} (n+1-k-j)\, \delta_j .$$

Regularity implies $(\lambda + \rho_{\mathfrak{b}}, \alpha) \neq 0$ for any $\alpha \in \Delta$, so this together with the $\mathfrak{b}_0$-dominancy of $\lambda + \rho_{\mathfrak{b}}$ shows that

$$\frac{2(\lambda + \rho_{\mathfrak{b}}, \alpha)}{(\alpha, \alpha)} > 0 \tag{10.28}$$

for all roots α of $\mathfrak{b}_{\bar{0}}$.

For each $\mathfrak{g}$, substituting $\alpha = \varepsilon_i - \varepsilon_{i+1}, \delta_i - \delta_{i+1}$ into (10.28) yields the conditions $a_i \geq a_{i+1}$, $b_i \geq b_{i+1}$, respectively. Indeed, we obtain $(\lambda, \varepsilon_i - \varepsilon_{i+1}) + (\rho_{\mathfrak{b}}, \varepsilon_i - \varepsilon_{i+1}) > 0$, $-(\lambda, \delta_i - \delta_{i+1}) - (\rho_{\mathfrak{b}}, \delta_i - \delta_{i+1}) > 0$ yielding $a_i - a_{i+1} + 1 > 0$, $b_i - b_{i+1} + 1 > 0$. Hence, $a_i \geq a_{i+1}, b_i \geq b_{i+1}$ since λ is an integral weight.

Now for $\mathfrak{g} = \mathfrak{osp}(m|2n)$ with $m \neq 2$ substituting $\alpha = 2\delta_n$ yields $b_n \geq \lfloor \frac{m}{2} \rfloor$. Indeed, we obtain $-(\lambda, \delta_n) - (\rho_{\mathfrak{b}}, \delta_n) \geq 0$ which reduces to $b_n + \frac{1}{2} - k > 0$ and $b_n + 1 - k > 0$ in the two respective cases. Then since $b_n, k \in \mathbb{Z}$, we get $b_n \geq k = \lfloor \frac{m}{2} \rfloor$. Hence, $b_n \geq \#\{i \mid a_i \neq 0\}$.

When $\mathfrak{g} = \mathfrak{osp}(2|2n)$, substituting $\alpha = 2\delta_n$ yields $b_n \geq 0$. Indeed, we obtain $-(\lambda, \delta_n) - (\rho_{\mathfrak{b}}, \delta_n) \geq 0$ which reduces to $b_n + 1 > 0$. Then since $b_n, k \in \mathbb{Z}$, we get $b_n \geq 0$.

If $\mathfrak{g} = \mathfrak{osp}(2k + 1|2n)$, substituting $\alpha = \varepsilon_n$ yields $a_n \geq 0$. Indeed, we obtain $(\lambda, \varepsilon_n) + (\rho_{\mathfrak{b}}, \varepsilon_n) > 0$ which reduces to $a_m + \frac{1}{2} > 0$. Thus, $a_m \geq 0$ since $a_m \in \mathbb{Z}$.

Finally, for $\mathfrak{g} = \mathfrak{osp}(2k|2n)$, substituting $\alpha = \varepsilon_{n-1} + \varepsilon_n$ yields $a_{n-1} \geq -a_n$. Indeed, we obtain $(\lambda, \varepsilon_{n-1} + \varepsilon_n) + (\rho_{\mathfrak{b}}, \varepsilon_{n-1} + \varepsilon_n) > 0$ which reduces to $a_{n-1} + a_n + 1 > 0$. Hence, $a_{n-1} \geq -a_n$ since $a_{n-1}, a_n \in \mathbb{Z}$. Combining with the above condition yields $a_{n-1} \geq |a_n|$. □

We now turn our attention to supergeometry. In the late nineteen seventies Manin constructed flag supermanifolds. Given a finite-dimensional superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, and a sequence of increasing dimensions

$$0 < a_1|b_1 < \cdots < a_n|b_n < \dim V,$$

Manin constructs the flag supermanifold $X = F\ell(a_1|b_1, \ldots, a_n|b_n, V)$. Its reduced manifold X_{red} equals the direct product

$$F\ell(a_1, \ldots, a_n, V_{\bar{0}}) \times F\ell(b_1, \ldots, b_n, V_{\bar{1}})$$

of usual flag manifolds, and the conormal bundle $\mathcal{N}^*_{X_{red}/X}$ in X is an $GL(n) \times GL(m)$-linearized bundle of rank equal to $\dim(\mathfrak{g}_{\bar{1}}/\mathfrak{b}_{\bar{1}})$. Manin constructs X as the representing object of a flag functor, and the uniqueness of such a representing object determines the structure of the sheaf $\mathcal{O}_X$ as a sheaf of $\mathbb{Z}_2$-graded rings. An important result of Skornyakov implies that $\mathcal{O}_X \simeq \Lambda^{\cdot}(\mathcal{N}^*_{X_{red}/X})$ only for very special sequences of dimensions $a_1|b_1, \ldots, a_n|b_n$ [P2]. In fact, Skornyakov showed that already if $\dim V = 2|2$, then $F\ell(1|1, V)$ is a nonsplit supermanifold [Man].

The flag supermanifold $X = F\ell(a_1|b_1, \ldots, a_n|b_n, V)$ is naturally endowed with a transitive action of the supergroup $GL(V)$ (the reader could give a definition of

such an action in the language of superschemes!), and moreover, the stabilizer in G of any point in X_{red} is a parabolic subsupergroup P of G. In fact, as it was noticed in [P2], the supermanifold X is the universal categorical quotient G/P.

For the Lie supergroups $G = SOSp(m|2n)$, $\mathbf{P}(n)$, $Q(n)$, the situation is similar. Following Manin, one considers flags of superspaces in a $\mathbb{Z}_2$-graded space V and defines flag supermanifolds of respectively isotropic (with respect to an even nondegenerate form on V for $SOSp(m|2n)$ and an odd nondegenerate form on V for $\mathbf{P}(n)$) flags, or Π-symmetric flags for $G = Q(n)$), see [Man, P2].

We now describe the connection of parabolic subsupergroups of G to flags of the natural representation V of G. First of all, a *representation* or a *module* of a Lie supergroup $G = (G', \mathfrak{g})$ is a $\mathfrak{g}$-module M that is also a G'-module such that both the induced structures of a $\mathfrak{g}_{\bar{0}}$-module on M coincide. Next, the supergroups $GL(m|n)$, $SOSp(m|2n)$, $\mathbf{P}(n)$, $Q(n)$ come equipped with a *defining* or *natural* representation V of respective dimension $m|n$, $m|2n$, $n|n$, $n|n$. Given a parabolic subsupergroup $P \subset G$, we can consider the socle filtration of V as a P-module, and this defines a $\mathbb{Z}_2$-graded flag in V which we associate to P. Conversely, given any $\mathbb{Z}_2$-graded flag F of V, we take $\mathfrak{p}$ in $\mathfrak{g}$ to be the (maximal) stabilizer of F, and P_{red} to be the parabolic subgroup of G_{red} corresponding to $\mathfrak{p}_{\bar{0}}$. The subsupergroup $(P_{red}, \mathfrak{p})$ is by definition the *stabilizer* of F.

In all cases, the data defining the flag supermanifolds are in 1-1 correspondence with conjugacy classes of parabolic subsupergroups P of G, and in what follows we write G/P for a flag supermanifold X such that P is the stabilizer of a point on X.

The definition of a G-linearized vector bundle on G/P is the same as in the case of a Lie group, see Sect. 10.1. We concentrate on such bundles of minimal rank. For $P = B$ and $G \neq Q(n)$ (i.e., for supermanifolds of maximal flags), this rank equals $1|0$ or $0|1$. Moreover, a G-linearized bundle $\mathcal{L}$ on G/B is determined uniquely by its restriction $\mathcal{L}_{red}$ to $(G/B)_{red}$, which in turn is determined by a finite-dimensional B_{red}-module. If $\lambda : B_{red} \to \mathbb{C}^\times$ is a character, then there is a G_{red}-linearized bundle $\mathcal{O}_{red}(\lambda)$ on $(G/B)_{red}$, and it defines a unique G-linearized vector bundle on G/B of rank $1|0$. In this presentation, we will identify $\mathcal{L}$ with $\Pi\mathcal{L}$, since the information we provide about the cohomology of $\mathcal{L}$ and $\Pi\mathcal{L}$ is essentially the same. For $G = Q(n)$, the situation is different as any character $\lambda : B_{red} \to \mathbb{C}^\times$ determines a unique representation $\boldsymbol{\lambda}$ of B defined up to parity change. The dimension of $\boldsymbol{\lambda}$ is given by (10.21). In all cases, there are uniquely defined G-linearized bundles $\mathcal{O}(\lambda)$ on G/B, which we choose to define here up to parity change. Finally, the reader may like to prove that the category of G-linearized vector bundles on G/P is canonically equivalent to the category of finite-dimensional P-modules.

10.2.2 Bott–Borel–Weil Theorem for Typical Weights

Theorem 10.7 ([P2]) *Let $G = GL(m|n)$, $SOSp(m|2n)$, $Q(n)$. Let B be a Borel subsupergroup of G and let $\lambda : B_{red} \to \mathbb{C}^\times$ be a character of B_{red} such that $\lambda - \rho_{\mathfrak{b}}$*

is a typical weight. If $\lambda - \rho_{\mathfrak{b}}$ is not regular, then

$$H^k(G/B, \mathcal{O}(\lambda)) = 0 \text{ for all } k \geq 0.$$

If $\lambda - \rho_{\mathfrak{b}}$ is regular, we have

$$H^k(G/B, \mathcal{O}(\lambda)) = \begin{cases} L_B(w(-\lambda + \rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}})^* \text{ for } k = \ell(w) \\ 0 \quad \text{for } k \neq \ell(w) \end{cases}$$

where $\ell(w)$ is the length, with respect to the simple roots of B_{red}, of the unique element $w \in W$ for which the weight $w(-\lambda + \rho_{\mathfrak{b}})$ is B_{red}-dominant.

Our proof follows in general the same idea as Demazure's proof from Sect. 10.1 but uses essential new ingredients. In the rest of this section λ is an integral weight.

We start with the following.

Lemma 10.8 (Demazure's Lemma) *Let $G = GL(m|n)$, $SOSp(m|2n)$, $\mathbf{P}(n)$. Let B a Borel subsupergroup and let α be a simple even root of B such that $\frac{1}{2}\alpha$ is not a root of $\mathfrak{g}$. Assume that $\frac{2(\lambda - \rho_{\mathfrak{b}}, \alpha)}{(\alpha, \alpha)} \leq 0$, then for any $k \geq 0$*

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k+1}(G/B, \mathcal{O}(w_\alpha(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}})),$$

where $w_\alpha \in W$ is the reflection at α.

Proof The proof is essentially the same as the proof of the isomorphism (10.12) since G/B is a $\mathbb{P}^1$-fibration over G/P where P is generated by B and by the $SL(2)$-subgroup corresponding to the root α. □

An obvious problem however is that for a given B as above, there may not be sufficiently many simple even roots as in Lemma 10.8 in order to extend the proof of Theorem 10.3 to Lie supergroups in a straightforward way. In fact, B may not have any even simple roots; therefore one may not even manage to get started.

The key remedy for this problem is the following.

Lemma 10.9 (Skornyakov's Odd Reflection Lemma) *Let $G = GL(m|n)$, $SOSp(m|2n)$, $\mathbf{P}(n)$. Let B_1 and B_2 be Borel subsupergroups of G connected by a nondegenerate odd reflection corresponding to an odd root α. Under the assumption that $(\lambda - \rho_{\mathfrak{b}_1}, \alpha) \neq 0$, there is a canonical isomorphism of G-modules*

$$H^k(G/B_1, \mathcal{O}(\lambda)) = H^k(G/B_2, \mathcal{O}(\lambda - \rho_{\mathfrak{b}_1} + \rho_{\mathfrak{b}_2}))$$

for any $k \geq 0$.

After reading the proof of Theorem 10.7 below, the reader will notice that Skornyakov's Lemma and Demazure's Lemma are sufficient to prove Theorem 10.7 for $G = GL(m|n)$. For $G = SOSp(m|2n)$, we need also the following analog of Lemma 10.8.

Lemma 10.10 *Let $G = SOSp(m|2n)$. Suppose α is a simple odd root of a Borel subsupergroup B of G such that 2α is a root. If $\frac{2(\lambda-\rho_{\mathfrak{b}},\alpha)}{(\alpha,\alpha)} \leq 0$, then for $k \geq 0$*

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k+1}(G/B, \mathcal{O}(w_{2\alpha}(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}})),$$

where $w_{2\alpha} \in W$ is the reflection at 2α.

Finally, for $G = Q(n)$ we have the following.

Lemma 10.11 *Let $G = Q(n)$, and let B be the Borel subsupergroup of G corresponding to the standard Borel subsuperalgebra from Definition 2.12. If $\lambda = (\lambda_1, \ldots, \lambda_{i'}, \lambda_{i'+1}, \ldots, \lambda_k)$ satisfies $\lambda_{i'} \leq \lambda_{i'+1}$, $\lambda_{i'} + \lambda_{i'+1} \neq 0$ for some i', then*

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k+1}(G/B, \mathcal{O}(\lambda'))$$

for $\lambda' = (\lambda_1, \ldots, \lambda_{i'+1}, \lambda_{i'}, \ldots, \lambda_k)$.

Lemmas 10.9–10.11 are proved with full detail in [P2]. The language used there is the language of the flag realizations of G/B, and it may take some time for the interested reader to compare the statements of Lemmas 10.9, 10.10 and 10.11 with Propositions 1, 3 and 4 in [P2], respectively.

Corollary 10.12 *Under the respective assumptions of Lemmas 10.8, 10.10, 10.11, suppose that*

- $w_\alpha(\lambda - \rho_{\mathfrak{b}}) = \lambda - \rho_{\mathfrak{b}}$ *in Lemma 10.8;*
- $w_{2\alpha}(\lambda - \rho_{\mathfrak{b}}) = \lambda - \rho_{\mathfrak{b}}$ *in Lemma 10.10*
- $\lambda_{i'} = \lambda_{i'+1}$ *in Lemma 10.11.*

Then

$$H^{\cdot}(G/B, \mathcal{O}(\lambda)) = 0.$$

Proof In each case by iterating the shift in cohomology claimed by the respective lemma, we see that

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{\dim(G/B)_{red}+1}(G/B, \mathcal{O}(\lambda)) = 0.$$

□

We are now ready for

Proof of Theorem 10.7 It suffices to show four facts:

(1) if $\lambda - \rho_{\mathfrak{b}}$ is not regular, equivalently, if $w_\alpha(\lambda - \rho_{\mathfrak{b}}) = \lambda - \rho_{\mathfrak{b}}$ for some reflection $w_\alpha \in W$, then $H^{\cdot}(G/B, \mathcal{O}(\lambda)) = 0$;
(2) if $\lambda - \rho_\beta$ is regular, then

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k-l(w)}(G/B', \mathcal{O}(w(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}'}))$$

for a unique $w \in W$ determined by the condition that the weight $w(\lambda - \rho_{\mathfrak{b}})$ is B_{red}-antidominant;

(3) if $\lambda - \rho_{\mathfrak{b}}$ is regular and B_{red}-antidominant, then

$$H^k(G/B, \mathcal{O}(\lambda)) = 0$$

for $k > 0$;

(4) if $\lambda - \rho_{\mathfrak{b}}$ is regular and B_{red}-antidominant, then

$$H^0(G/B, \mathcal{O}(\lambda)) = L_B(-\lambda)^*.$$

To prove (1), we fix a root α which is not odd and that corresponds to a reflection $w_\alpha \in W$ such that $w_\alpha(\lambda - \rho_{\mathfrak{b}}) = \lambda - \rho_{\mathfrak{b}}$. Then we choose a chain of odd reflections $B = B_1, \ldots, B_r$ where B_r is a Borel subsupergroup for which α or $-\alpha$ is a simple root of B_r in the case $G = GL(m|n)$, $Q(n)$, and respectively $\frac{\alpha}{2}$ or $-\frac{\alpha}{2}$ is a simple root of B_r in the case $G = OSp(m|2n)$. Such a chain exists by Lemma 2.11. Then by applying Lemma 10.9 to all pairs B_i, B_{i+1}, we conclude that

$$H^{\cdot}(G/B, \mathcal{O}(\lambda)) = H^{\cdot}(G/B_r, \mathcal{O}(\lambda - \rho_{\mathfrak{b}} + \rho_{\mathfrak{b}_r}))$$

where $\mathfrak{b}$ and $\mathfrak{b}_r$ are the Lie superalgebras of B and B_r. Now Corollary 10.12 yields

$$H^{\cdot}(G/B_r, \mathcal{O}(\lambda - \rho_{\mathfrak{b}} + \rho_{\mathfrak{b}_r})) = 0.$$

Consequently, $H^{\cdot}(G/B, \mathcal{O}(\lambda)) = 0$.

Next we prove (3). Consider all Borel subsupergroups B' with $B'_{red} = B_{red}$, and recall that for any such B', there exists a chain of odd reflections $B = B_1, ..., B_s = B'$. Moreover,

$$H^{\cdot}(G/B, \mathcal{O}(\lambda)) = H^{\cdot}(G/B', \mathcal{O}(\lambda - \rho_{\mathfrak{b}} + \rho_{\mathfrak{b}'})).$$

Now we represent the element $w_0 \in W$, which is the longest with respect to the simple roots $\alpha_1, \ldots, \alpha_t$ of B_{red}, as a product of simple reflections $w_{\alpha_1}, \ldots, w_{\alpha_t}$. For each of these simple roots α_i, there is B' such that either α_i or $\alpha_i/2$ is a simple root of B'. Application of the appropriate choice among Lemmas 10.8, 10.10, 10.11 yields

$$\begin{aligned} H^k(G/B, \mathcal{O}(\lambda)) &= H^k(G/B', \mathcal{O}(\lambda - \rho_{\mathfrak{b}} + \rho_{\mathfrak{b}'})) \\ &= H^{k+1}(G/B', \mathcal{O}(w_{\alpha_i}(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}'})) \\ &= H^{k+1}(G/B, \mathcal{O}(w_{\alpha_i}(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}})). \end{aligned}$$

The B_{red}-antidominancy of $\lambda - \rho_{\mathfrak{b}}$ allows us to iterate this isomorphism along the decomposition of w_0 as a product of reflections, and obtain at the end

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k-l(w_0)}(G/B, \mathcal{O}(w_0(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}})).$$

Since $l(w_0) = \dim(G/B)_{red}$, we conclude that

$$H^k(G/B, \mathcal{O}(\lambda)) = 0$$

for $k > 0$.

We now turn to (2). Here $\lambda - \rho_{\mathfrak{b}}$ is regular but without the assumption of antidominancy on $\lambda - \rho_{\mathfrak{b}}$. We consider a chain of adjacent Borel subsupergroups $B = B_1, \ldots, B_q$, such that the pair B_{q-1}, B_q corresponds to a root α which is not odd and such that the weight $w_\alpha(-\lambda + \rho_{\mathfrak{b}})$ is in a chamber closer to the B_{red}-antidominant chamber in $\mathfrak{h}_{\bar{0}}^*$.

Then by Lemma 10.9 and the appropriate choice of Lemmas 10.8, 10.10, 10.11, we have

$$H^k(G/B, \mathcal{O}(\lambda)) = H^{k-1}(G/B'_q, \mathcal{O}(w_\alpha(\lambda - \rho_{\mathfrak{b}}) + \rho_{\mathfrak{b}_q}),$$

and a straightforward induction argument on the length of w implies our conclusion (3). The case when $\lambda - \rho_{\mathfrak{b}}$ is B_{red}-antidominant is the base of induction.

It remains to show (4), i.e., that $H^0(G/B, \mathcal{O}(\lambda)) = L_B(-\lambda)^*$ holds whenever $\lambda - \rho_{\mathfrak{b}}$ is regular and B_{red}-antidominant. Note that under these assumptions, $L_B(-\lambda)$ is a simple finite dimensional G-module by Lemma 10.6 applied to $-\lambda$. We now need to refer to the Kac–Weyl character formula from [K3] for $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, as well as to a character formula proved by the first author in [P1], which give the formal character of $L_B(-\lambda)$ under the assumption that $-\lambda + \rho_{\mathfrak{b}}$ is B_{red}-dominant and regular. Indeed, using the dimension formula for $L_B(-\lambda)$ resulting from these character formulas, one can check that the Euler characteristic of $\mathcal{O}(\lambda)$ coincides with $\dim L_B(-\lambda)$. Next, the homomorphism dual to the injective homomorphism of B-modules $\mathbb{C}_{-\lambda} \to L_B(-\lambda)$ shows that there is a surjective homomorphism of B-modules $L_B(-\lambda)^* \to \mathbb{C}_\lambda$, i.e., that λ is the B-lowest weight of $L_B(-\lambda)^*$.

On the other hand, $H^0(G/B, \mathcal{O}(\lambda)) \neq 0$ due to the Euler characteristic argument and the vanishing of the higher cohomology of $\mathcal{O}(\lambda)$. Therefore the evaluation morphism of G-bundles

$$H^0(G/B, \mathcal{O}(\lambda)) \otimes \mathcal{O}_{G/B} \to \mathcal{O}(\lambda) \tag{10.29}$$

induces a surjective homomorphism of B-modules

$$H^0(G/B, \mathcal{O}(\lambda)) \to \mathbb{C}_\lambda. \tag{10.30}$$

Since λ is the B-lowest weight of $L_B(-\lambda)^*$, we conclude via Frobenius reciprocity (which remains valid in our setting!) that the simple module $L_B(-\lambda)^*$ is isomorphic

to a quotient of $H^0(G/B, \mathcal{O}(\lambda))$. Finally, the equality

$$\dim H^0(G/B, \mathcal{O}(\lambda)) = \dim L_B(-\lambda)^*$$

implies $H^0(G/B, \mathcal{O}(\lambda))$ and $L_B(-\lambda)^*$ are canonically isomorphic. □

Next, let's discuss briefly the case $G = \mathbf{P}(n)$. If $B \subset \mathbf{P}(n)$ is the standard Borel subsupergroup, the result for $H^{\cdot}(G/B, \mathcal{O}(\lambda))$ is simply that

$$H^{l(w)}(G/B, \mathcal{O}(\lambda)) = L_B(w(-\lambda + \rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}})^* \tag{10.31}$$

under the assumption that there is a unique $w \in W$ such that the weight $w(-\lambda+\rho_{\mathfrak{b}})$ is B_{red}-dominant, and that

$$H^k(G/B, \mathcal{O}(\lambda)) = 0$$

for all other choices of k and λ. Note that here the $\mathfrak{g}_{\bar{0}}$-regularity of $\lambda-\rho_{\mathfrak{b}}$ is sufficient for (10.31). If B' is an arbitrary Borel subsupergroup such that $B'_{red} = B_{red}$, then B' has a well-defined *height* $h_{B'}$: this is the number of indices l for which $2\varepsilon_l$ is not a root of B'. It is proved in [P2] that under the stronger assumption that $|\mu_i - \mu_j| \neq 0, 1, 2, 3$ for $\mu = \lambda - \rho_{\mathfrak{b}'}$, the only nonzero cohomology group of $\mathcal{O}(\lambda)$ on G/B' is $H^{l(w)}(G/B', \mathcal{O}(\lambda))$ for the unique element $w \in W$ such that the weight $w(\lambda - \rho_{\mathfrak{b}'})$ is B'_{red}-dominant. However, a major difference from all cases considered above is that this cohomology group is no longer an irreducible G-module and has length $2^{h_{B'}}$. The simple constituents of $H^{l(w)}(G/B', \mathcal{O}(\lambda))$ are isomorphic to $L'_B(w(-\lambda + \sum_{j\in F(\mathfrak{b}')} 2\varepsilon_j + \rho_{\mathfrak{b}'}) - \rho_{\mathfrak{b}'})^*$, where $F(\mathfrak{b}')$ runs over all finite subsets of roots of the form $2\varepsilon_j$ that are not roots of B'. This new effect is the result of the existence of degenerate odd reflections. For details and a proof of the result, see [P2].

The reduction from G/B to G/P doesn't work in the case of supergroups for the simple reason that if a G-linearized bundle E on G/P has the form $p_*\mathcal{O}(\lambda)$ where $p : G/B \to G/P$ is the natural morphism, then the higher direct images $R^i\pi_*\mathcal{O}(\lambda)$ no longer have to vanish. Therefore the Leray spectral sequence argument from Theorem 10.5 does not apply [PS1]. Nevertheless, in [PS1] the cohomology of a generic bundle $\mathcal{O}_{G/P}(\lambda)$ has been computed. Let $B \subset P$ be respectively a fixed Borel subsupergroup and a parabolic subsupergroup of $G = GL(m|n), OSp(m|2n), Q(n)$. We consider only P-adapted weights λ, i.e., integral weights for which $\mathbb{C}_\lambda$, or $\boldsymbol{\lambda}$ for $G = Q(n)$ is a P-module. Then the G-linearized bundle $\mathcal{O}_{G/P}(\lambda) = p_*\mathcal{O}_{G/B}(\lambda)$ is well defined. The main result of [PS1] claims that for a Zariski open set of P-adapted weights λ, the cohomology $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$ vanishes unless $k = l(w)$ where $w \in W$ is the unique element for which the weight $w(-\lambda + \rho_{\mathfrak{b}_{\bar{0}}})$ is B_{red}-dominant. Moreover, $H^{l(w)}(G/P, \mathcal{O}_{G/P}(\lambda))$ is a simple G-module whose highest weight is computed in [PS1]. For $G = \mathbf{P}(n)$, the situation is similar to the case $P = B$. Namely, if h_P is the number of weights of the form $2\varepsilon_j$ that are not roots of $\mathbf{P}(n)$, then $H^{l(w)}(G/P, \mathcal{O}_{G/P}(\lambda))$ has length 2^{h_P} while $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$ vanishes for $k \neq l(w)$, see [PS1].

Some further results of Bott–Borel–Weil type for Lie supergroups can be found in [PS2] and in the work of K. Coulembier [Cou].

10.3 Toward a Bott-Borel-Weil Theorem for Locally Reductive Ind-Groups

Similar to the case of Lie supergroups, for ind-groups G, the classical Bott–Borel–Weil Theorem motivates a study of the relationship between rational G-modules and the cohomology of G-linearized sheaves of the homogeneous spaces of G. In the case of ind-groups, such sheaves can be vector bundles of finite rank or they can be projective limits of vector bundles of growing rank.

In [DPW], both of the above cases have been studied, and this section can serve as an introduction to the ideas of this work. For the technical parts, we refer the reader to [DPW]. The theory cannot yet be considered complete, and at the end of the section we discuss some open problems.

10.3.1 The Case of a Line Bundle

Let G be a connected locally reductive ind-group, i.e., an inductive limit of connected reductive algebraic groups G_n for $n \in \mathbb{Z}_{>0}$. Without loss of generality, we can assume that the defining morphisms $\varphi_n : G_n \to G_{n+1}$ are injective, hence from now on we suppose that we have fixed inclusions $G_n \subset G_{n+1}$. Let $P_n \subset G_n$ be parabolic subgroups with the property that $P_{n+1} \cap G_n = P_n$, and let P be the direct limit, or union of the groups P_n. Then the Lie algebra $\mathfrak{p} = \mathrm{Lie}\, P$ is a *splitting* parabolic subalgebra of $\mathfrak{g}$, i.e., it contains a splitting Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ of the form $\mathfrak{h} = \bigcup_n \mathfrak{h}_n$, where $\mathfrak{h}_n \subset \mathfrak{p}_n$ are Cartan subalgebras. In the rest of this chapter we assume the exhaustion $G = \bigcup_n G_n$ to be fixed, and we refer to an ind-subgroup P of the form $\bigcup_n P_n$ as above as to a *splitting parabolic subgroup*. If each $P_n = B_n$ is a Borel subgroup of G_n, then $P = B = \bigcup B_n$ is by definition a *splitting Borel subgroup* of G. If $P = \bigcup_n P_n$ is a splitting parabolic subgroup, we have closed embeddings of homogeneous spaces

$$G_n/P_n \subset G_{n+1}/P_{n+1}. \tag{10.32}$$

We denote the union $\bigcup_n G_n/P_n$ (via the embeddings (10.32)) by G/P, or by G/B if $P = B$ is a splitting Borel subgroup of G.

If G is one of the classical ind-groups, for instance if $G = GL(\infty)$, the interpretation of a splitting parabolic subgroup $P \subset G$ as the stabilizer of a generalized flag $\mathcal{F}_P$, compatible with a fixed basis, enables us to give an explicit interpretation of G/P in this case. More precisely, if $\mathcal{F}_P$ is the unique $\mathfrak{p}$-stable

generalized flag in the natural $\mathfrak{g}$-module (here $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{p} = \operatorname{Lie} P$), then G/P is identified with the G-orbit $G \cdot \mathcal{F}_P$. The structure of an ind-variety on $G \cdot \mathcal{F}_P$ comes from presenting $G \cdot \mathcal{F}_P$ as a union $\bigcup_n G_n/P_n$, and in [DP2] an explicit presentation of $G \cdot \mathcal{F}_P$ as a union of usual flag varieties is given. The same applies to the classical ind-groups $SO(\infty)$, $Sp(\infty)$ where one has to consider isotropic generalized flags compatible with the basis of respective type B, D or C, see Sect. 5.2.4. We recommend the reader to look up the details in [DP2].

Now let more generally, $X = \bigcup_n X_n$ be any ind-variety defined via a system of inclusions $X_n \subset X_{n+1}$ of algebraic varieties. A (*finite-rank*) *vector bundle* E on X is by definition a system of vector bundles E_n on X_n such that $E_{n+1|X_n} = E_n$ for all $n \in \mathbb{Z}_{>0}$. This implies that all E_n have equal finite rank, called the *rank* of E. In particular, a *line bundle* $\mathscr{L}$ on X is a vector bundle on X of rank 1.

We start our generalization of the Bott–Borel–Weil Theorem by considering line bundles on the ind-varieties G/P. Note that by Proposition 10.2 each line bundle on G_n/P_n admits a G_n-linearization. It is therefore clear that any line bundle $\mathscr{L}$ on G/P admits a G-linearization corresponding to a character $\lambda : P \to \mathbb{C}^\times$, which is nothing but a projective system of characters $\lambda^n : P_n \to \mathbb{C}^\times$ (note that λ^n is not the n-th power of λ). We write $\mathscr{L} = \mathcal{O}_{G/P}(\lambda)$. In what follows, we will consider a projective system of characters $\lambda = \{\lambda^n\}$ as a projective system of G_n-integral weights $\{\lambda^n \in \mathfrak{h}_n^*\}$, where $\mathfrak{h}_n$ are fixed nested Cartan subalgebras of $\mathfrak{g}_n$.

We are interested in computing the total cohomology $H^\cdot(G/P, \mathcal{O}_{G/P}(\lambda))$ for any λ, in particular $H^\cdot(G/P, \mathscr{L})$ for any line bundle $\mathscr{L}$ on G/P. This problem has two aspects: one is to prove a vanishing theorem for all cohomology groups $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$ except for possibly one value of k, and the second is to then compute the single nonvanishing cohomology group $H^{k_0}(G/P, \mathcal{O}_{G/P}(\lambda))$ as a "G-module". Strictly speaking, by a (*rational*) *G-module* we mean a direct limit $\varinjlim M_n$ of rational finite-dimensional G_n-modules M_n. Clearly, any G-module is countable dimensional, and it turns out that a cohomology group $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$ is never a G-module unless $\lambda = 0$. However, such a cohomology group is a *pro-rational G-module*, which by definition is the (projective) limit $\varprojlim N_n$ of a projective system

$$\cdots \to N_n \to N_{n-1} \to \cdots \to 0$$

of finite-dimensional rational G_n-modules.

A (rational) G-module is *locally simple* if it is the direct limit of simple G_n-modules M_n under injective maps $M_n \to M_{n+1}$. In what follows, we will assume that the maps $M_n \to M_{n+1}$ are fixed inclusions $M_n \subset M_{n+1}$, and we write $M = \bigcup_n M_n$ for a locally simple G-module.

In a similar way we define a P-module, a locally simple P-module and a pro-rational P-module.

Recall now that $\mathcal{O}_{G/P}(\lambda) = \varprojlim \mathcal{O}_{G_n/P_n}(\lambda^n)$, and moreover that the Bott–Borel–Weil Theorem (more precisely Theorem 10.5) enables us to compute $H^k(G/P, \mathcal{O}_{G_n/P_n}(\lambda^n))$ for any k, n and λ^n. Therefore in order to compute $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$, we need to compare $H^k(G/P, \varprojlim \mathcal{O}_{G_n/P_n}(\lambda^n))$ with the

projective limit of cohomologies $\varprojlim H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n))$ for a fixed $k \in \mathbb{Z}_{\geq 0}$. There is a rather general statement which implies that in fact these two vector spaces are always equal. Indeed, we can consider each sheaf $\mathcal{O}_{G_n/P_n}(\lambda^n)$ as a sheaf on the topological space $G/P = \varinjlim G_n/P_n$ just by extending $\mathcal{O}_{G_n/P_n}(\lambda^n)$ by zero outside of G_n/P_n. Then the universality of the projective limit

$$\varprojlim H^k(G/P, \mathcal{O}_{G_n/P_n}(\lambda^n))$$

implies the existence of a natural morphism of G-modules

$$\gamma : H^k(G/P, \varprojlim \mathcal{O}_{G_n/P_n}(\lambda^n)) \to \varprojlim H^k(G/P, \mathcal{O}_{G_n/P_n}(\lambda^n)).$$

Moreover, a general result, see [Har, Theorem 4.5], [GD, O_{III}, 13.3.1], implies that γ is an isomorphism for any k and any choice of λ^n. More precisely, this result applies to an inverse limit $\varprojlim \mathcal{A}_n$ of arbitrary sheaves $\mathcal{A}_n$ of $\mathcal{O}_X$-modules (where in our case $X = G/P$) and claims that a sufficient condition for γ to be an isomorphism is that the vector space $H^k(X, \mathcal{A}_n)$ be finite dimensional for each n. In our case this condition is satisfied as $H^k(G/P, \mathcal{O}_{G_n/P_n}(\lambda^n)) = H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n))$ is a finite-dimensional space (G_n/P_n is compact and $\mathcal{O}_{G_n/P_n}(\lambda^n)$ is a coherent $\mathcal{O}_{G_n/P_n}$-module). In this way we have sketched the proof of the following more general result.

Proposition 10.13 *Let $\{\mathcal{A}_n\}$ be a projective system of coherent $\mathcal{O}_{G_n/P_n}$-modules on the direct limit $G/P = \varinjlim G_n/P_n$, i.e., $\mathcal{A}_{n+1|G_n/P_n} = \mathcal{A}_n$. In particular, $\mathcal{A} := \varprojlim \mathcal{A}_n$ may be a vector bundle of finite rank on G/P or simply a line bundle. Then there is a natural isomorphism*

$$\gamma : H^k(G/P, \mathcal{A}) \xrightarrow{\sim} \varprojlim H^k(G_n/P_n, \mathcal{A}_n). \tag{10.33}$$

If each $\mathcal{A}_n$ is G_n-linearized, the right-hand side of (10.33) (and hence also its left-hand side) is a pro-rational G-module.

We recommend the reader to reconstruct the complete proof of Proposition 10.13 by using the original sources [Har] and EGA [GD].

Our next observation is that $\varprojlim H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)) \neq 0$ if and only there exists $w \in W$ with $k = l(w)$ and such that $w(-\lambda^n + \rho^n) - \rho^n$ is B_n-dominant for sufficiently large n (here ρ^n stands here for the half-sum of the roots of a fixed Borel subgroup $B_n \subset P_n$, cf. Theorem 10.5, and the condition of B_n-dominancy was introduced in Sect. 10.1.2). Then $H^l(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)) = 0$ for all $l \neq k$, and this implies immediately that

$$H^k(G/P, \mathcal{O}_{G/P}(\lambda)) = \varprojlim H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n))$$

is nonzero for at most one k (λ being fixed). Therefore the problem of computing the cohomology $H^{\cdot}(G/P, \mathcal{O}_{G/P}(\lambda))$ reduces to finding, for each $k \in \mathbb{Z}_{\geq 0}$, an explicit

criterion in terms of P and λ for the nonvanishing of $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$, and then computing $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$ as a pro-rational G-module. We will discuss this problem in more detail below.

In what follows, given a splitting Borel subgroup $B = \bigcup_n B_n$, we call a character $\lambda = \{\lambda^n\}$ *B-dominant* (respectively, *B-antidominant*) if λ^n is B_n-dominant (respectively, B_n-antidominant) for all $n \in \mathbb{Z}_{>0}$.

Let us first consider the case when λ is antidominant with respect to any splitting Borel subgroup $B \subset P$. This is equivalent to saying that there is a unique locally simple G-module $L_B(\lambda)$ (respectively $L_B(-\lambda)$) such that the 1-dimensional B-module $\mathbb{C}_\lambda$ (respectively $\mathbb{C}_{-\lambda}$) is a B-submodule of $L_B(\lambda)$ (respectively, $L_B(-\lambda)$). It is easy to prove that as a module over the Lie algebra of G, $L_B(\lambda)$ is the unique simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ (respectively, $L_B(-\lambda)$ is the unique simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-\lambda}$).

We claim now that if the character λ of P is B-antidominant for some splitting Borel subgroup $B \subset P$ (equivalently, for any splitting Borel subgroup of P), there is a canonical isomorphism of pro-rational P-modules $H^0(G/P, \mathcal{O}_{G/P}(\lambda)) \simeq L_B(-\lambda)^*$. Recall that $H^0(G/P, \mathcal{O}_{G/P}(\lambda))$ is equal to the projective limit

$$\cdots \to H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)) \to H^0(G_{n-1}/P_{n-1}, \mathcal{O}_{G_{n-1}/P_{n-1}}(\lambda^{n-1})) \to \cdots . \tag{10.34}$$

Moreover, the restriction maps

$$H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)) \to H^0(P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)_{|P_n}) = \mathbb{C}_{-\lambda^n}$$

together with the maps (10.34) form a commutative diagram

$$\begin{array}{ccccccc} \ldots \to & H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)) & \to & H^0(G_{n-1}/P_{n-1}, \mathcal{O}_{G_{n-1}/P_{n-1}}(\lambda^{n-1})) & \to & \ldots \\ & \downarrow & & \downarrow & & \\ \ldots \xrightarrow{\sim} & \mathbb{C}_{\lambda^n} & \xrightarrow{\sim} & \mathbb{C}_{\lambda^{n-1}} & \xrightarrow{\sim} & \ldots, \end{array} \tag{10.35}$$

where the map $\mathbb{C}_{\lambda^n} \xrightarrow{\sim} \mathbb{C}_{\lambda^{n-1}}$ in the lower row is an isomorphism of 1-dimensional P_{n-1}-modules (recall that the character λ^n of P_n restricts to the character λ^{n-1} of P_{n-1}). The dual diagram

$$\begin{array}{ccccccc} \ldots \leftarrow & H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n))^* & \leftarrow & H^0(G_{n-1}/P_{n-1}, \mathcal{O}_{G_{n-1}/P_{n-1}}(\lambda^{n-1}))^* & \leftarrow & \ldots \\ & \uparrow & & \uparrow & & \\ \ldots \xleftarrow{\sim} & \mathbb{C}_{-\lambda^n} & \xleftarrow{\sim} & \mathbb{C}_{-\lambda^{n-1}} & \xleftarrow{\sim} & \ldots, \end{array} \tag{10.36}$$

together with the Bott–Borel–Weil Theorem (Theorem 10.5) yields a commutative diagram

$$\begin{array}{ccccccc} \dots & \longleftarrow & L_{B_n}(-\lambda^n) & \longleftarrow & L_{B_{n-1}}(-\lambda^{n-1}) & \longleftarrow & \dots \\ & & \uparrow & & \uparrow & & \\ \dots & \xleftarrow{\sim} & \mathbb{C}_{-\lambda^n} & \xleftarrow{\sim} & \mathbb{C}_{-\lambda^{n-1}} & \xleftarrow{\sim} & \dots \end{array} . \tag{10.37}$$

This latter diagram shows that the direct limit $\varinjlim L_B(-\lambda^n)$ in the upper row of (10.37) is isomorphic to the highest weight G-module $L_B(-\lambda)$; hence the inverse limit $H^0(G/P, \mathcal{O}_{G/P}(\lambda))$ of the upper row of the original diagram (10.35) is nothing but $L_B(-\lambda)^*$.

In this way we sketched the proof of the following proposition.

Proposition 10.14 *If $G = \bigcup_n G_n$ is a connected locally reductive ind-group, $P \subset G$ is a splitting parabolic subgroup, $B \subset G$ is a splitting Borel subgroup, and the projective system of weights $\lambda = \{\lambda^n\}$ defines a 1-dimensional P-module, then*

$$H^k(G/P, \mathcal{O}_{G/P}(\lambda)) \neq 0$$

for at most one value of $k \in \mathbb{Z}_{\geq 0}$. Furthermore, $H^0(G/P, \mathcal{O}_{G/P}(\lambda)) \neq 0$ if and only if λ is B-antidominant. In the latter case there is an isomorphism of pro-rational G-modules

$$H^0(G/P, \mathcal{O}_{G/P}(\lambda)) = L_B(-\lambda)^*. \tag{10.38}$$

□

We now come to the interesting problem of computing the potentially nonzero higher cohomology group $H^k(G/P, \mathcal{O}_{G/P}(\lambda))$ in the case when

$$H^0(G/P, \mathcal{O}_{G/P}(\lambda)) = 0.$$

We start with a key special case which suggests the answer in greater generality. More specifically, we assume that G is root-reductive. By definition this means that the embeddings of Lie algebras $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ induced by our fixed embeddings of algebraic groups $G_n \subset G_{n+1}$ are root embeddings. Next, we assume that $P = B$ is a union of Borel subgroups B_n such that every simple root of B_n is also a simple root of B_{n+1}, i.e., the corresponding Borel subalgebra $\mathfrak{b}$ is Dynkin (see Example 5.6). In this case we have a well-defined Weyl group W, which is simply the inductive limit of the Weyl groups W_n under the root embeddings $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ induced by the embeddings $G_n \subset G_{n+1}$. Moreover, each element $w \in W$ has a well-defined length defined in terms of the simple roots of $\mathfrak{b} = \mathrm{Lie}\, B$.

Note that if $H^k(G/B, \mathcal{O}_{G/B}(\lambda)) \neq 0$ then, for large enough n, we have $k = l(w_n)$ where $w_n(-\lambda^n + \rho^n)$ is $\mathfrak{b}_n$-dominant. This yields well-defined elements $w_n \in$

W_n with $l_n(w_n) = l(w_n) = k$ (here $l_n(\cdot)$ stands for length with respect to the simple roots of $\mathfrak{b}_n$ and $l(\cdot)$ denotes length with respect to the simple roots of $\mathfrak{b}$). The equality $l(w_n) = l(w_{n+1})$ (holding for $n >> 0$) implies that for some $n_0 \in \mathbb{Z}_{>0}$, all simple reflections which enter the respective decompositions of w_n for $n \geq n_0$ are among a finite number of simple reflections w_{α_k}, $k = 1,\ldots, s$ for some s. Now fix n such that $\alpha_1,\ldots, \alpha_s$ are simple roots of B_n and consider the four regular weights $-\lambda^n + \rho^n$, $-\lambda^{n+1} + \rho^{n+1}$, $w_n(-\lambda^n + \rho^n)$, $w_{n+1}(-\lambda^{n+1} + \rho^{n+1})$. By definition,

$$(-\lambda^{n+1} + \rho^{n+1})_{|\mathfrak{h}_n} = -\lambda^n + \rho^n$$

and, according to the Bott–Borel–Weil Theorem, the weights $w_{n+1}(-\lambda^{n+1} + \rho^{n+1})$ and $w_n(-\lambda^n + \rho^n)$ are nothing but respective dominant representatives of $-\lambda^{n+1} + \rho^{n+1}$ and $-\lambda^n + \rho^n$. Therefore,

$$w_{n+1}(-\lambda^{n+1} + \rho^{n+1})_{|\mathfrak{h}_n} = w_n(-\lambda^n + \rho^n). \tag{10.39}$$

Since in addition $w_{n+1} \in W_n$, the Eq. (10.39) implies immediately that $w_{n+1} = w_n$ as elements of W_n. This yields the conclusion that $w_n = w_{n+1}$ for any $n \geq n_0$, i.e., the elements w_n determine a well-defined element $w \in W$.

Furthermore, it is easy to check (the reader should do this in full detail) that for any $n \geq n_0$ and $\mathcal{Q}_n := \mathcal{O}_{G_n/B_n}(w(\lambda^n - \rho^n) + \rho^n)$, there is a commutative diagram

$$\begin{array}{ccc} H^0(G_{n+1}/B_{n+1}, \mathcal{Q}_{n+1}) & \simeq & H^k(G_{n+1}/B_{n+1}, \mathcal{O}_{G_{n+1}/B_{n+1}}(\lambda^{n+1})) \\ \downarrow & & \downarrow \\ H^0(G_n/B_n, \mathcal{Q}_n) & \simeq & H^k(G_n/B_n, \mathcal{O}_{G_n/B_n}(\lambda^n)) \end{array} \tag{10.40}$$

where the rows of (10.40) are the Demazure isomorphisms (10.13) established in Sect. 10.1 and the columns are the morphisms on cohomology induced by the restriction morphisms

$$\mathcal{O}_{G_{n+1}/B_{n+1}}(\lambda^{n+1}) \to \mathcal{O}_{G_n/B_n}(\lambda^n),$$

$$\mathcal{Q}_{n+1} = \mathcal{O}_{G_{n+1}/B_{n+1}}(w(\lambda^{n+1} - \rho^{n+1}) + \rho^{n+1}) \to \mathcal{O}_{G_n/B_n}(w(\lambda^n - \rho^n) + \rho^n) = \mathcal{Q}_n.$$

Let Δ^+ denote the set of roots of $\mathfrak{b}$. The diagram (10.40) implies the existence of a canonical isomorphism

$$H^k(G/B, \mathcal{O}_{G/B}(\lambda)) \simeq H^0\left(G/B, \mathcal{O}_{G/B}\left(w(\lambda) - \sum_{\alpha \in \Delta^+, w(\alpha) \notin \Delta^+} \alpha\right)\right)$$

(note that the projective system of weights $-w(\rho^{n+1}) + \rho^{n+1}$ corresponds to the weight $-\sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha$ considered as a weight of $\mathfrak{g}$). Since

$$H^0\left(G/B, \mathcal{O}_{G/B}(w(\lambda) - \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha)\right) \simeq L_B\left(-w(\lambda) + \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha\right)^*$$

by (10.40), we obtain finally that $H^k(G/B, \mathcal{O}_{G/B}(\lambda)) \neq 0$ implies the existence of a unique $w \in W$ such that $w(\lambda) - \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha$ is a B-antidominant regular weight of $\mathfrak{g}$, and the existence of a G-module isomorphism

$$H^k(G/B, \mathcal{O}_{G/B}(\lambda)) \simeq L_B\left(-w(\lambda) + \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha\right)^*.$$

Conversely, given a line bundle $\mathcal{O}_{G/B}(\lambda)$ and an element $w \in W$ such that $w(\lambda) - \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha$ is a B-antidominant weight, by using the diagram (10.40), we show that

$$H^{l(w)}(G/B, \mathcal{O}_{G/B}(\lambda)) \simeq L_B\left(-w(\lambda) + \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+}\alpha\right)^* \neq 0.$$

In order to state a more general result, we need to introduce another notion. Let B be any splitting Borel subgroup of a connected root-reductive ind-group G and let $\mathfrak{b} \subset \mathfrak{g}$ be its Lie algebra. Then, according to Chap. 5, $\mathfrak{b}$ has a set of simple roots which in general will not span the root lattice Δ. Nevertheless, if an element $w \in W$ can be decomposed as a product of simple reflections, it has a well-defined length: this is the length of the shortest such expression, or equivalently the length of w as an element of W_n for $n >> 0$. In what follows we will simply say that w has a *well-defined length with respect to* B.

We come to the first central result in this section.

Theorem 10.15 *Let $G = \bigcup_n G_n$ be a connected root-reductive ind-group, let P be a splitting parabolic subgroup, let $B \subset P$ be a splitting Borel subgroup and let $\lambda = \{\lambda^n\}$ be a character of P. Then*

$$H^{\cdot}(G/P, \mathcal{O}_{G/P}(\lambda)) \neq 0$$

if and only if

$$H^{k_0}(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n)) \neq 0$$

for some fixed $k_0 \in \mathbb{Z}_{>0}$ and $n >> 0$. This condition is equivalent to the existence of $w \in W$ of well-defined length $l(w)$ with respect to B, and such that $w(\lambda) - \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+} \alpha$ is a B-antidominant weight. Furthermore, in the latter case we have $l(w) = k_0$,

$$H^k(G/P, \mathcal{O}_{G/P}(\lambda)) = 0 \text{ for } k \neq l(w), \tag{10.41}$$

and there is an isomorphism of pro-rational G-modules

$$H^{l(w)}(G/P, \mathcal{O}_{G/P}(\lambda)) \simeq L_B\left(-w(\lambda) + \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+} \alpha\right)^*. \tag{10.42}$$

Sketch of Proof We have already outlined the proof of this theorem in the special case when the Lie algebra $\mathfrak{b}$ is a Dynkin Borel subalgebra. Consider now the case when $P = B$ is an arbitrary splitting Borel subgroup. Here we need the following crucial statement which is a particular case of the main combinatorial result in [DPW] (see Proposition 9.7). For the proof, we refer the reader to [DPW] and recommend that the reader attempts to do at least some special cases on his/her own.

Lemma 10.16 *Denote by $l_n(\cdot)$ the length function on W_n corresponding to the Borel subgroup $B_n \subset G_n$. Assume $\lambda = \{\lambda^n\}$ satisfies $l_n(w_n) = k$ for a fixed k and for $n >> 0$, where w_n is the unique element in W such that $w_n(\lambda^n - \rho^n)$ is B_n-antidominant. Then for $n >> 0$, the elements w_n define an element $w \in W$ of well-defined length k with respect to B, such that $w(\lambda) - \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+} \alpha$ is a well-defined B-antidominant weight.*

The assumption of Lemma 10.16 is equivalent to the condition

$$H^k(G_n/B_n, \mathcal{O}_{G_n/B_n}(\lambda^n)) \neq 0$$

for $n >> 0$. Moreover, Lemma 10.16 allows us to extend the above argument from the case when $\mathfrak{b}$ is a Dynkin Borel subalgebra to the case of any splitting Borel subgroup B, establishing in this way the isomorphism (10.42). The vanishing statement (10.41) follows then immediately from Proposition 10.14.

Conversely, if we are given $w \in W$ as in the statement of Theorem 10.15, the above argument applies and again yields (10.42) and (10.41).

In the case of a general splitting parabolic subgroup P, we notice that the nonvanishing of $H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda^n))$ for $n >> 0$ is equivalent to that of $H^k(G_n/B_n, \mathcal{O}_{G_n/B_n}(\lambda^n))$ (Theorem 10.5). Therefore we can apply the Theorem to G/B and obtain in particular an isomorphism of pro-rational G-modules

$$H^k(G/B, \mathcal{O}_{G/B}(\lambda)) \simeq L_B\left(-w(\lambda) + \sum_{\alpha\in\Delta^+, w(\alpha)\notin\Delta^+} \alpha\right)^*. \tag{10.43}$$

Note however that for each n, there is a canonical isomorphism

$$H^k(G_n/B_n, \mathcal{O}_{G_n/B_n}(\lambda)) = H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(\lambda)) \tag{10.44}$$

induced by the projection $p_n : G_n/B_n \to G_n/P_n$, see (10.19). Since the following diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\cup & & \cup \\
G_n/B_n & \xrightarrow{p_n} & G_n/P_n \\
\cup & & \cup \\
G_{n-1}/B_{n-1} & \xrightarrow{p_{n-1}} & G_{n-1}/P_{n-1} \\
\cup & & \cup \\
\vdots & & \vdots
\end{array} \tag{10.45}$$

is commutative, we see immediately that the isomorphisms (10.43) and (10.44) yield (10.42). The vanishing statement (10.41) is now also obvious. □

10.3.2 The Case of Growing Rank

Theorem 10.15 extends the classical Bott–Borel–Weil Theorem concerning line bundles to root-reductive ind-groups. A next natural task is to find an analog of Theorem 10.5 for root-reductive ind-groups. Here the situation is much more complicated and is not yet completely understood.

We start by defining analogs of the bundles $\mathcal{O}_{G/P}(E)$ from Sect. 10.1. In the case of an ind-group, the natural choice is to fix a generally infinite-dimensional locally simple (rational) P-module $E = \bigcup_n E_n$, and then consider the G-linearized sheaf $\mathcal{O}_{G/P}(E^*)$. Now by definition, $\mathcal{O}_{G/P}(E^*)$ is the inverse limit of the obvious diagram of restriction maps of vector bundles

$$\cdots \to \mathcal{O}_{G_{n+1}/P_{n+1}}(E_{n+1}^*) \to \mathcal{O}_{G_n/P_n}(E_n^*) \to \cdots, \tag{10.46}$$

and in this sense resembles a vector bundle. In particular, Proposition 10.13 applies to any sheaf of the form $\mathcal{O}_{G/P}(E^*)$ and yields isomorphisms

$$H^k(G/P, \mathcal{O}_{G/P}(E^*)) \xrightarrow{\sim} \varprojlim H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) \tag{10.47}$$

for all k.

In principle it would be interesting to consider also the G-linearized sheaf $\mathcal{O}_{G/P}(E)$ defined as the inverse limit of the diagram of restriction maps

$$\cdots \to \mathcal{O}_{G_{n+1}/P_{n+1}}(E) \to \mathcal{O}_{G_n/P_n}(E) \to \cdots$$

where $\mathcal{O}_{G_n/P_n}(E)$ are reducible G-linearized vector bundles of infinite rank on G_n/P_n. However, Proposition 10.13 does not apply to $\mathcal{O}_{G/P}(E)$ (a concrete example can be found in [DPW]), and in general the structure of the cohomology $H^\cdot(G/P, \mathcal{O}_{G/P}(E))$ is not understood.

In the rest of this chapter, we study the sheaves $\mathcal{O}_{G/P}(E^*)$. The relation to the line bundles studied so far is obvious: $\mathcal{O}_{G/P}(\lambda) = \mathcal{O}_{G/P}((\mathbb{C}_{-\lambda})^*)$, where $\mathbb{C}_{-\lambda}$ is the 1-dimensional P-module of weight $-\lambda$ (or equivalently the P-module dual to the P-module $\mathbb{C}_\lambda$).

We start with the observation that since each vector bundle $\mathcal{O}_{G_n/P_n}(E_n^*)$ has at most one nonzero cohomology group (Theorem 10.5), the projective limit

$$H^k(G/P, \mathcal{O}_{G/P}(E^*)) = \varprojlim H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*))$$

is nonzero for at most one value of $k \in \mathbb{Z}_{>0}$. Furthermore, the isomorphism (10.47) has also a general analog in our setting. To see this, we first need a definition. We call the simple P-module $E = \bigcup_n E_n$ *dominant*, if for every $n >> 0$ there is a simple finite-dimensional G_n-module L_n such that E_n is a simple P_n-submodule of L_n. Note that L_n is determined up to isomorphism, since L_n is isomorphic to the unique simple quotient of the induced module $U(\mathfrak{g}_n) \otimes_{U(\mathfrak{p}_n)} E_n$ for $\mathfrak{p}_n = \mathrm{Lie}\, P_n$. In what follows we denote L_n by $L_{P_n}(E_n)$. If each E_n is 1-dimensional, i.e., $E_n = \mathbb{C}_{\lambda^n}$, then E is dominant if and only if the projective system of weights $\{\lambda^n\}$ yields a P-dominant character (weight).

Note now that if $H^0(G/P, \mathcal{O}_{G/P}(E^*)) = \varprojlim H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) \neq 0$, then the sequence (10.46), together with the isomorphism (10.47) for each n, yields a commutative diagram

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) & \longrightarrow & H^0(G_{n-1}/P_{n-1}, \mathcal{O}_{G_{n-1}/P_{n-1}}(E_{n-1}^*)) & \longrightarrow & \ldots \\
 & & \downarrow \simeq & & \downarrow \simeq & & \\
\ldots & \longrightarrow & L_{P_n}(E_n)^* & \longrightarrow & L_{P_{n-1}}(E_{n-1})^* & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & E_n^* & \longrightarrow & E_{n-1}^* & \longrightarrow & \ldots
\end{array} \tag{10.48}$$

By dualizing (10.48), we obtain

$$\begin{array}{ccccccc}
\ldots & \leftarrow & H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*))^* & \leftarrow & H^0(G_{n-1}/P_{n-1}, \mathcal{O}_{G_{n-1}/P_{n-1}}(E_{n-1}^*))^* & \leftarrow & \ldots \\
 & & \uparrow \simeq & & \uparrow \simeq & & \\
\ldots & \leftarrow & L_{P_n}(E_n) & \leftarrow & L_{P_{n-1}}(E_{n-1}) & \leftarrow & \ldots \\
 & & \uparrow & & \uparrow & & \\
\ldots & \leftarrow & E_n & \leftarrow & E_{n-1} & \leftarrow & \ldots
\end{array} \tag{10.49}$$

The middle row of (10.49) determines a (rational) G-module of the form $\varinjlim L_{P_n}(E_n)$, and the lowest row shows the existence of an injection of P-modules

$$E \to \varinjlim L_{P_n}(E_n).$$

As there is clearly a unique (up to isomorphism) G-module $L_P(E)$ with the above properties, we obtain that E is dominant and that there is an isomorphism of pro-rational G-modules

$$H^0(G/P, \mathcal{O}_{G/P}(E^*)) = \varprojlim H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) \simeq L_P(E)^*. \tag{10.50}$$

Conversely, if E is dominant, we reverse the argument and show that

$$H^0(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$$

and that there is an isomorphism (10.50).

In this way we arrive at the following analog of Proposition 10.14.

Proposition 10.17 *For any connected locally reductive ind-group $G = \bigcup_n G_n$, for any splitting parabolic subgroup $P \subset G$, and for any (rational) locally simple P-module E, we have*

$$H^k(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$$

for at most one value of $k \in \mathbb{Z}_{>0}$. Furthermore,

$$H^0(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$$

if and only if $H^0(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) \neq 0$ for all $n > 0$, or equivalently, if and only if E is dominant. In the latter case there is an isomorphism of pro-rational G-modules

$$H^0(G/P, \mathcal{O}_{G/P}(E^*)) \simeq L_P(E)^*.$$

We ask next what is the analog of the isomorphism (10.41) in our setting. First of all, we have to assume again that G is root-reductive. Then the situation is as follows: there is a particular case for E in which the answer is quite easy and natural, and there is the general case in which the answer is quite remarkable but less easy. Understanding the latter case is the main achievement of the paper [DPW], and here we will only sketch the result.

Recall that in the finite-dimensional case, $\mathcal{O}_{G/P}(E^*) \simeq p_*\mathcal{O}_{G/B}(-\lambda)$, where λ is the B-highest weight of E. In the infinite-dimensional case, the same holds under the assumption that E has a highest weight. More precisely, assume that there is a splitting Borel subgroup B of G with $B \subset P$ and such that E is a B-highest weight module. (This holds automatically if E is finite dimensional.) Let the highest weight of E be λ. As it is easy to check, this means that

$$\mathcal{O}_{G/P}(E^*) \simeq p_*\mathcal{O}_{G/B}(-\lambda), \tag{10.51}$$

where

$$p : G/B \to G/P$$

is the natural projection of ind-varieties, i.e., the commutative diagram (10.51) of natural projections and closed embeddings which we discussed above. Indeed, $\mathcal{O}_{G/P}(E^*)$ is defined by the projective system $\{\mathcal{O}_{G_n/P_n}(E_n^*)\}$, and $p_*\mathcal{O}_{G/B}(-\lambda)$ is defined by the projective system $\{(p_n)_*\mathcal{O}_{G_n/B_n}(-\lambda^n)\}$. However, according to the proof of Theorem 10.5, there are natural isomorphisms

$$(p_n)_*\mathcal{O}_{G_n/B_n}(-\lambda^n) \simeq \mathcal{O}_{G_n/P_n}(E_n^*), \tag{10.52}$$

and these isomorphisms are clearly compatible with the projective systems $p_*\mathcal{O}_{G/B}(-\lambda)$ and $\mathcal{O}_{G/P}(E^*)$ (to the reader: prove this latter statement). Hence (10.51) induces a natural isomorphism (10.52) as claimed.

The isomorphism (10.51) enables us to immediately compute the cohomology $H^\cdot(G/P, \mathcal{O}_{G/P}(E^*))$. Indeed, first of all we note that

$$H^\cdot(G/P, \mathcal{O}_{G/P}(E^*)) = H^\cdot(G/B, \mathcal{O}_{G/B}(-\lambda)). \tag{10.53}$$

The canonical isomorphism (10.53) is a direct corollary of the fact that

$$H^\cdot(G/P, \mathcal{O}_{G/P}(E^*)) = \varprojlim H^\cdot(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)),$$

$$H^\cdot(G/B, \mathcal{O}_{G/B}(-\lambda)) = \varprojlim H^\cdot(G_n/B_n, \mathcal{O}_{G_n/B_n}(-\lambda^n)),$$

and that the Leray spectral sequence applied to the horizontal arrows in (10.45) yields a system of canonical isomorphisms

$$H^\cdot(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) = H^\cdot(G_n/B_n, \mathcal{O}_{G_n/B_n}(-\lambda^n))$$

compatible with the vertical arrows in (10.45). Therefore, computing the cohomology $H^{\cdot}(G/P, \mathcal{O}_{G/P}(E^*))$ is the same as computing $H^{\cdot}(G/B, \mathcal{O}_{G/B}(-\lambda))$, and the latter is given by Theorem 10.15.

In this way we arrive at the following analog of Theorem 10.15.

Proposition 10.18 *Let G, B and P be as in Theorem 10.15. Assume in addition that $E = \bigcup_n E_n$ is a locally simple rational P-module such that the inclusions $E_n \subset E_{n+1}$ map the B_n-highest weight space of E_n into the B_{n+1}-highest weight space of E_{n+1}. Then, if $\lambda = \{\lambda^n\}$ is the corresponding projective system of highest weights, we have*

$$H^{\cdot}(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$$

if and only if there is $w \in W$ such that $w(\lambda) - \sum_{\alpha \in \Delta^+, w(\alpha) \notin \Delta^+} \alpha$ is a B-antidominant weight. Furthermore, in the latter case,

$$H^l(G/P, \mathcal{O}_{G/P}(E^*)) = 0 \quad \text{for } l \neq l(w)$$

and there is an isomorphism of pro-rational G-modules

$$H^{l(w)}(G/P, \mathcal{O}_{G/P}(E^*)) \simeq L_B(-w(\lambda) + \sum_{\alpha \in \Delta^+, w(\alpha) \notin \Delta^+} \alpha)^*.$$

We now come to the problem of computing $H^{\cdot}(G/P, \mathcal{O}_{G/P}(E^*))$ under the assumption that the P-module E has no highest weight with respect to any splitting Borel subgroup $B \subset P$. In this case we can ask whether the unique nonvanishing cohomology group $H^k(G/P, \mathcal{O}_{G/P}(E^*))$ (in case it exists) is isomorphic to $L_P(F)^*$ for some simple P-module F. As the following example shows, the answer to this question is "no".

Example 10.19 Let $G = GL(\infty)$ with the standard upper left-hand corner exhaustion $GL(\infty) = \bigcup_n GL(n)$, $\mathfrak{g} = \mathfrak{gl}(\infty)$, let $\mathfrak{h}$ be the diagonal Cartan subalgebra and let $\mathfrak{p} \subset \mathfrak{h}$ be the maximal parabolic subalgebra with roots $\{\varepsilon_1 - \varepsilon_i \mid 2 \leq i\} \cup \{\varepsilon_i - \varepsilon_j \mid 2 \leq i \neq j\}$. Let P be the corresponding parabolic subgroup, let B be the upper triangular Borel subgroup, and let E_n be the simple finite-dimensional P_n-module with B_n-highest weight $\lambda^n = -\varepsilon_1 + n\varepsilon_2$. The branching rule for the inclusion $\mathfrak{gl}(n) \subset \mathfrak{gl}(n+1)$ ensures that E_{n+1} has a unique $\mathfrak{gl}(n)$-submodule isomorphic to E_n (to the reader: recall the Gelfand–Zetlin branching rule [Zh], and verify this claim). Therefore the union $\bigcup E_n$ is a well-defined simple P-module. By applying Theorem 10.5 to $\mathcal{O}_{G_n/P_n}(E_n^*)$, we compute that

$$H^1(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) \simeq L_n^*$$

where L_n is the simple finite-dimensional $GL(n)$-module with B_n-highest weight $(n-1)\varepsilon_1$. This makes it clear that

$$H^1(G/P, \mathcal{O}_{G/P}(E^*)) \simeq L^*$$

where L is the simple module $S^\infty_{\mathbb{A}} V$ defined by the sequence $\mathbb{A} = \{1, 2, 3, \dots\}$, see Sect. 8.3.3.

Let us show that L has no simple P-submodule, and in fact no simple P'-submodule for any proper splitting parabolic subgroup P' whose Lie algebra $\mathfrak{p}'$ contains $\mathfrak{h}$. It suffices to consider L as a module over $\mathfrak{g} = \mathfrak{gl}(\infty)$. Recall that as a $\mathfrak{gl}(n)$-module, L_n is nothing but the $(n-1)$th symmetric power of the natural representation V_n of $\mathfrak{gl}(n)$. Fix a parabolic subalgebra $\mathfrak{p}' \supset \mathfrak{h}$, and let $\mathfrak{b}' \subset \mathfrak{p}'$ be a Borel subalgebra corresponding to a strict linear order $\prec$ on $\mathbb{Z}_{>0}$. Any nonzero zero $\mathfrak{p}'$-submodule of L contains the $\mathfrak{b}'_n$-highest vectors l_n of L_n: this follows from the observation that the socle of L_n as a $\mathfrak{b}'_n$-module is spanned by l_n. The $\mathfrak{h}$-weight of l_n has the form

$$\mu_n = \sum_{k \in \mathbb{Z}_{\geq 0}} \varepsilon_k + (n-2)\varepsilon_{j(n)} - \varepsilon_{k_1(n)} - \dots - \varepsilon_{k_{n-2}(n)}$$

for some $j(n) \prec k_t(n)$ where $k_1(n), \dots, k_{n-2}(n)$ are pairwise distinct. In addition, if $n_1 < n_2$, then $j(n_2) \prec j(n_1)$ or $j(n_1) = j(n_2)$. Note that for any fixed $j(n)$ all possible indices k with $j(n) \prec k$ will appear as $k_t(n)$ for a suitable n. Therefore, having fixed n_1 we can choose n_2 so that $k_{n_1-2}(n_1) \prec k_{n_2-2}(n_2)$.

Now if L' is a simple $\mathfrak{p}'$-submodule of L, then the weights $\mu_{n_1} - \mu_{n_2}$ for any $n_1 < n_2$ have to be weights of $U(\mathfrak{p}')$. Thus, with a choice of n_1, n_2 as above, we see that $-\varepsilon_r + \varepsilon_{k_{n_2-2}(n_2)}$ must be a root of $\mathfrak{p}'$ for some $r \prec k_{n_2-2}(n_2)$, for arbitrarily large $k_{n_2-2}(n_2)$ in terms of the strict linear ordering $\prec$. However, if $\mathfrak{p}' \neq \mathfrak{g}$, there will always be a k_0 such that no roots of the form $-\varepsilon_r + \varepsilon_{k_0}$ for $r \prec k_0$ are roots of $\mathfrak{p}'$. Therefore the condition that L admits a proper simple $\mathfrak{p}'$-submodule $L' \subset L$ implies $\mathfrak{p}' = \mathfrak{g}$. Consequently, L has no simple $\mathfrak{p}'$-submodule for a proper parabolic subalgebra $\mathfrak{p}' \supset \mathfrak{h}$. ◯

It might appear that we have arrived at a dead-end as the G-module L in the above example seems to have little to do with the parabolic subgroup P, and hence in general we would have no control over the higher cohomology groups of $\mathcal{O}_{G/P}(E^*)$. However, the main result of [DPW] proposes a way out of this seeming dead-end and explains the new infinite-dimensional phenomenon we are encountering. More precisely, it is proved in [DPW] that if $H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)) \neq 0$ for some fixed k and all $n >> 0$, then there is a splitting Borel subgroup $B = \bigcup_n B_n$ and an element $w \in W$ of length k with respect to B such that

$$H^k(G/P, \mathcal{O}_{G/P}(E^*)) = L_{wP}({}^wE)^* \tag{10.54}$$

for a certain parabolic subgroup ${}^wP$ and for some unique simple ${}^wP$-module ${}^wE$. To define ${}^wP$ it suffices to define its Lie algebra ${}^w\mathfrak{p} := \mathrm{Lie}\,{}^wP$. We fix a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{p}$, and define ${}^w\mathfrak{p}$ to be the minimal parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}$ and all root spaces of the form $\mathfrak{g}^\beta$ where both $\mathfrak{g}^{w^{-1}(\beta)}$ and $\mathfrak{g}^{-w^{-1}(\beta)}$ are root spaces of $\mathfrak{p}$. Clearly, in general the inclusion $P \subset {}^wP$ is proper while the inclusion ${}^wP \subset G$ is not always proper. For instance, if P is the parabolic subalgebra from the above example, one checks immediately that ${}^wP = G$. Therefore in this particular example ${}^wE = L$.

The least obvious part in the above statement is the existence of w. After that, the construction of ${}^wE$ is clear in principle as each simple $\mathfrak{g}_n$-module $H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(E^*))^*$ contains a unique simple $({}^w\mathfrak{p}) \cap \mathfrak{g}_n$-submodule ${}^wE_n$. We leave it to the reader to verify that ${}^wE_n \subset {}^wE_{n+1}$ (one can also look it up in [DPW]). Therefore we have a well-defined simple ${}^wP$-module ${}^wE$ such that (10.54) holds. To summarize, we give a formal statement of the main result. A more general statement is proved in [DPW].

Theorem 10.20 *Let G and P be as in Theorem 10.15 and let $E = \bigcup_n E_n$ be any locally simple rational P-module. Then $\mathcal{O}_{G/P}(E^*)$ has at most one nonvanishing cohomology group. If*

$$H^k(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0,$$

there is a splitting Borel subgroup $B \subset P$, and a unique element $w \in W$ of length k with respect to B, and a unique locally simple ${}^wP$-module ${}^wE$ such that there is an isomorphism of pro-rational G-modules

$$H^k(G/P, \mathcal{O}_{G/P}(E^*)) \simeq L_{{}^wP}({}^wE)^*.$$

Clearly, Theorem 10.20 is a direct generalization of Theorem 10.5, but there is no highest weight in its statement as the G-module $L_{{}^wP}({}^wE)$ is in general not a highest weight module.

Let's mention that the main result of [DPW], and in particular Theorem 10.20, was extended from root-reductive ind-groups to diagonal ind-groups in [DP5]. In addition, we would like to point out that an analytic approach to the Bott–Borel–Weil Theorem for direct limit groups was developed in [NRW].

10.4 Open Problems

Here are three suggestions for further exploration.

1. Extend Theorem 10.15 to arbitrary locally reductive ind-groups. The main statement here would be that if G is not diagonal, then generically $\mathcal{O}_{G/B}(\lambda)$ has no nonzero cohomology groups. The challenge is to describe the exceptional weights λ for which $H^\cdot(G/B, \mathcal{O}_{G/B}(\lambda)) \neq 0$.

2. Describe intrinsically the class of rational G-modules appearing as the direct limit $\varinjlim(H^k(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)))^*$ in the context of Theorem 10.20. In particular, is it true that any locally simple weight G-module appears in this way (here G is root-reductive)?
3. Study the homogeneous ind-spaces G/P (in particular G/B) where G is root-reductive but P is a parabolic subgroup which is no longer splitting. A new phenomenon here is that these ind-spaces are no longer locally proper. Find interesting phenomena related to these ind-spaces.

References

[AP] Arnal, D., Pinczon, G.: On algebraically irreducible representations of the Lie algebra $\mathfrak{sl}(2)$. J. Math. Phys. **15**, 350–359 (1974).

[BS] Bahturin, Yu., Strade, H.: Locally finite-dimensional simple Lie algebras (in Russian). Mat. Sb. **185**(2), 3–32 (1994).

[Ba] Baranov, A.: Finitary simple Lie algebras. J. Algebra **219**, 299–329 (1999).

[BaZ] Baranov, A., Zhilinski, A.: Diagonal direct limits of simple Lie algebras. Comm. Algebra **27**, 2749–2766 (1998).

[BB] Beilinson, A., Bernstein, J.: Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris **292**, 15–18 (1981).

[BGS] Beilinson, A., Ginzburg, V., Soergel, V.: Koszul duality patterns in representation theory. J. Amer. Math. Soc. **9**, 473–527 (1996).

[Be] Berezin, F. A.: Introduction to Superanalysis. Springer, Netherlands (1987).

[Bl] Block, R.: The irreducible representations of the Lie algebra $\mathfrak{sl}(2)$ and of the Weyl algebra. Adv. Math. **39**(1), 69–110 (1981).

[BW] Borel, A., Weil, A., (exposé by Serre, J.-P.): Representations linéaires et espaces homogènes Kählerians des groupes de Lie compacts (in French). Séminaire Bourbaki **2**(100), 447–454 (1954).

[Bo] Bott, R.: Homogeneous vector bundles. Ann. of Math. **66**(2), 203–248 (1957).

[B] Bourbaki, N.: Lie groups and Lie algebras, Chapters I, II-III, IV-VI, VII-VII. Masson 1971, 1972, 1981, 1998.

[BL] Britten, D. J., Lemire, F. W.: A classification of simple Lie modules having a 1-dimensional weight space. Trans. Amer. Math. Soc. **299**, 683–697 (1987).

[Cal] Calixto, L.: On the structure of simple bounded weight modules of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. Comm. Algebra **48**(10), 4256–4280 (2020).

[CalP] Calixto, L., Penkov, I.: Integrable bounded weight modules for classical Lie superalgebras at infinity. J. Pure Appl. Algebra **226**(2), (2022).

[Ch] Cheng, S.-J.: Differentiably simple Lie superalgebras and representations of semisimple Lie superalgebras. J. Algebra **173**(1), 1–43 (1995).

[ChW] Cheng, S., Wang, W.: Dualities and Representations of Lie Superalgebras. Grad. Stud. Math. Vol. 144, Providence, RI (2012).

[CP1] Chirvasitu, A., Penkov, I.: Representation categories of Mackey Lie algebras as universal monoidal categories. Pure Appl. Math. Q **13**, 77–121 (2017).

[CP2] Chirvasitu, A., Penkov, I.: Ordered tensor categories and representations of the Mackey Lie algebra of infinite matrices. Algebr. Represent. Theory **22**, 249–279 (2019).

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7

[CPS] Cline, E., Parshall, B., Scott, L.: Finite-dimensional algebras and highest weight categories. J. Reine Angew. Math. **391**, 85–99 (1988).
[Cou] Coulembier, K.: Bott-Borel-Weil theory and Bernstein–Gelfand reciprocity for Lie superalgebras. Transform. Groups **21**(3), 681–723 (2016).
[CouP] Coulembier, K., Penkov, I.: On an infinite limit of BGG categories $\mathcal{O}$. Moscow Math. J. **19**, 655–693 (2019).
[D] Dan-Cohen, E.: Borel subalgebras of root-reductive Lie algebras. J. Lie Theory **18**(1), 215–241 (2008).
[DP] Dan-Cohen, E., Penkov, I.: Parabolic and Levi Subalgebras of Finitary Lie Algebras. Int. Math. Res. Not. IMRN **6**, 1062–1101 (2010).
[DPS] Dan-Cohen, E., Penkov, I., Serganova, V.: A Koszul category of representations of finitary Lie algebras. Adv. Math. **289**, 250–278 (2016).
[DPSn] Dan-Cohen, E., Penkov, I., Snyder, N.: Cartan subalgebras of root-reductive Lie algebras. J. Algebra **308**, 583–611 (2007).
[De1] Demazure, M.: Une démonstration algébrique d'un théorème de Bott (in French). Invent. Math. **5**, 349–356 (1968).
[De2] Demazure, M: A very simple proof of Bott's theorem. Invent. Math. **33**, 271–272 (1976).
[DMP] Dimitrov, I., Mathieu, O., Penkov, I.: On the structure of weight modules. Trans. Amer. Math. Soc. **352**(6), 2857–2869 (2000).
[DP1] Dimitrov, I., Penkov, I.: Weight modules of direct limit Lie algebras. Int. Math. Res. Not. IMRN **5**, 223–249 (2000).
[DP2] Dimitrov, I., Penkov, I.: Ind-varieties of generalized flags as homogeneous spaces for classical ind-groups. Int. Math. Res. Not. IMRN **55**, 2935–2953 (2004).
[DP3] Dimitrov, I., Penkov, I.: Borel subalgebras of $\mathfrak{gl}(\infty)$. Resenhas IME-USP **6**, 111–119 (2004).
[DP4] Dimitrov, I., Penkov, I.: Locally semisimple and maximal subalgebras of the finitary Lie algebras $\mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$. Journ. of Algebra **322**, 2069–2081 (2009).
[DP5] Dimitrov, I., Penkov, I.: A Bott-Borel-Weil Theorem for diagonal ind-groups. Canad. J. Math. **63**(6), 1307–1327 (2011).
[DPW] Dimitrov, I., Penkov, I., Wolf, J. A.: A Bott-Borel-Weil Theorem for direct limits of algebraic groups, Amer. J. Math. **124**(5), 955–998 (2002).
[Dix] Dixmier, J.: Enveloping Algebras, Grad. Stud. Math., Vol. 11, Providence, RI (1996).
[DFO] Drozd, Yu. A., Futorny, V. M., Ovsienko, S. A.: Harish-Chandra subalgebras and Gelfand-Zetlin modules. In: Finite-dimensional algebras and related topics, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 424, pp. 79–93. Kluwer Acad. Publ., Dordrecht (1994).
[Dyn] Dynkin, E. B.: Semisimple subalgebras of semisimple Lie algebras (in Russian). Mat. Sb. **2**, 349–462 (1972).
[E] Enright, T. J.: On the algebraic construction and classification of Harish-Chandra modules. Proc. Nat. Acad. Sci. U.S.A. **75**(3) 1063–1065 (1978).
[EW] Enright, T. J., Wallach, N. R.: Notes on homological algebra and representations of Lie algebras. Duke Math. J. **47**, 1–15 (1980).
[EHS] Entova-Aizenbud, I., Hinich, V., Serganova, V.: Deligne categories and the limit of categories $\mathrm{Rep}(GL(m|n))$. IMRN **15**, 4602–4666 (2020).
[FGGr] Ferguson, T., Gorelik, M., Grantcharov, D.: Bounded highest weight modules of $\mathfrak{osp}(1, 2n)$. Proc. Sympos. Pure Math. **92**, 135–143 (2016).
[F] Fernando, S. L.: Lie algebra modules with finite-dimensional weight spaces. Trans. Amer. Math. Soc. **322**, 2857–2869 (1990).
[FPS] Frenkel, I., Penkov, I., Serganova, V.: A categorification of the boson-fermion correspondence via representation theory of $\mathfrak{sl}(\infty)$. Comm. Math. Phys. **341**(3), 911–931 (2016).
[FH] W. Fulton, J. Harris, Representation Theory. Grad. Texts Math. **129**, Springer-Verlag, New York (2004).
[FO] Futorny, V., Ovsienko, S.: Fibers of characters in Gelfand-Tsetlin categories. Trans. Amer. Math. Soc. **366**(8), 4173–4208 (2014).

[GelP] Gelfand, I., Ponomarev, V.: The category of Harish-Chandra modules over the Lie algebra of the Lorentz group (in Russian). Dokl. Akad. Nauk SSSR **176**, 243–246 (1967).
[GGr1] Gorelik, M., Grantcharov, D.: Bounded highest weight modules of $\mathfrak{q}(n)$. Int. Math. Res. Not. IMRN **22**, 6111–6154 (2014).
[GGr2] Gorelik, M., Grantcharov, D.: Simple bounded highest weight modules of basic classical Lie superalgebras. Transform. Groups (2020). https://doi.org/10.1007/s00031-020-09616-x.
[Gr1] Grantcharov, D.: Coherent families of weight modules of Lie superalgebras and an explicit description of the simple admissible $\mathfrak{sl}(n+1|1)$-modules. J. Algebra **265**, 711–733 (2003).
[Gr2] Grantcharov, D.: Explicit realizations of simple weight modules of classical Lie superalgebras. Cont. Math. **499** 141–148 (2009).
[GrP] Grantcharov, D., Penkov, I.: Simple bounded weight modules of $\mathfrak{sl}(\infty)$, $\mathfrak{sp}(\infty)$, $\mathfrak{o}(\infty)$. Transform. Groups **25**, 1125–1160 (2020).
[GH] Griffiths, P., Harris, J.: Principles of algebraic geometry. John Wiley and Sons (2014).
[GD] Grothendieck, A., Dieudonné, J. A.: Éléments de géométrie algébrique, EGA (in French). Institut des Hautes Études Scientifiques - Ihes (1960).
[Har] Hartshorne, R.: Algebraic geometry. Grad. Texts Math. **52**, Springer-Verlag, New York (1977).
[HS] Hausner, M., Schwartz, J.: Lie groups, Lie algebras. Gordon and Breach Science Publishers, New York - London - Paris (1968).
[Hel] Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Grad. Stud. Math., Vol. 34, Providence (2001).
[HTW] Howe, R., Tan, E., Willenbring, J.: Stable branching rules for classical symmetric pairs. Trans. Amer. Math. Soc. **357**(4), 1601–1626 (2005).
[Hoy] Hoyt, C.: Weight modules for $D(2,1,\alpha)$. In: Gorelik, M., Papi, P. (eds.) Advances in Lie superalgebras, Springer INdAM Ser. no. 7, pp. 91–100. Springer, Cham (2014).
[HoyPS] Hoyt, C., Penkov, I., Serganova, V.: Integrable $\mathfrak{sl}(\infty)$-modules and Category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$. J. Lond. Math. Soc. **99**(2), 403–427 (2019).
[K1] Kac, V. G.: Automorphisms of finite order of semisimple Lie algebras. Funktsional. Anal. i Prilozhen **3**, 94–96 (1969).
[K2] Kac, V. G.: Lie superalgebras. Adv. Math. **26**, 8–96 (1977).
[K3] Kac, V. G.: Representations of classical Lie superalgebras. Lecture Notes in Math. **676**, 597–626 (1978).
[K4] Kac, V. G.: Constructing groups associated to infinite-dimensional Lie algebras. In: Kac, V. (ed.) Infinite-dimensional groups with applications. Math. Sci. Res. Inst. Publ., Vol. 4, pp. 167–216. Springer, New York (1985).
[K5] Kac, V. G.: Infinite dimensional Lie algebras (3rd ed.). Cambridge: Cambridge University Press. (1990).
[Kn] Knapp, A. W.: Lie Groups beyond an Introduction. 2nd. ed., Progr. Math. Vol. 140, Birkhäuser, Boston (2002).
[KnVo] Knapp, A. W., Vogan, Jr., D. A.: Cohomological induction and unitary representations. Princeton University Press, Princeton, NJ (1995).
[Ko1] Kostant, B.: Lie algebra cohomology and the generalized Borel–Weil theorem. Ann. of Math. **74**, 329–387 (1961).
[Ko2] Kostant, B.: Graded manifolds, graded Lie theory and prequantization. In: Bleuler, K., Reetz, A. (eds.) Differential Geometrical Methods in Mathematical Physics. Lecture Notes in Math., Vol. 570. Springer, Berlin, Heidelberg (1977).
[Ko3] Kostant, B.: On Whittaker Vectors and Representation Theory. Invent. Math. **48**(2), 101–184 (1978).
[La] Lang, S.: Algebra. Grad. Texts Math. **211**, Springer-Verlag New York (2002).
[L] Leites, D. A.: Introduction to the theory of supermanifolds. Russian Math. Surveys **35**(1), 1–64 (1980).

[Mac] Mackey, G.: On infinite-dimensional linear spaces. Trans. Amer. Math. Soc. **57**, 155–207 (1945).

[Mal] Malcev, A. I.: On semisimple subgroups of Lie groups. Amer. Math. Soc. Transl. (1) 9, 172–213 (1962).

[Man] Manin, Yu.: Gauge Field Theory and Complex Geometry. Springer-Verlag, Berlin Heidelberg (1997).

[Mat] Mathieu, O.: Classification of irreducible weight modules. Ann. Inst. Fourier (Grenoble) **50**, 537–592 (2000).

[Maz] Mazorchuk, V.: On Categories of Gelfand-Zetlin Modules. In: Duplij S., Wess J. (eds) Noncommutative Structures in Mathematics and Physics. NATO Science Series (Series II: Mathematics, Physics and Chemistry), Vol. 22. Springer, Dordrecht (2001).

[Mu] Musson, I. M.: Lie superalgebras and enveloping algebras. Grad. Stud. Math. **131**, Providence, RI (2012).

[Nam] Nampaisarn, T.: Categories $\mathcal{O}$ for Dynkin Borel Subalgebras of Root-Reductive Lie Algebras. Dissertation, Jacobs University Bremen, Germany (2017).

[NRW] Natarajan, L., Rodrigues-Carrington, E., Wolf, J. A.: The Bott-Borel-Weil Theorem for direct limit Lie groups. Trans. Amer. Math. Soc. **353** 4583–4622 (2001).

[NP] Neeb, K.-H., Penkov, I.: Cartan subalgebras of $\mathfrak{gl}(\infty)$. Canad. Math. Bull. **46**, 597–616 (2003).

[O] Oseledets, C.: Root direct limits of Lie superalgebras. Dissertation, University of California, Riverside, U.S.A. (2001).

[P1] Penkov, I.: Characters of typical irreducible finite dimensional $q(m)$-modules (in Russian). Funktsional. Anal. i Prilozhen. **20**(1), 37–45 (1986).

[P2] Penkov, I.: Borel-Weil-Bott theory for classical Lie superalgebras. J. Math. Sci. **51**(1), 2108–2140 (1990).

[P3] Penkov, I.: Characters of strongly generic irreducible Lie superalgebra representations. Internat. J. Math. **9**, 331–366 (1998).

[P4] Penkov, I.: Classically semisimple locally finite Lie superalgebras. Forum Math. **16**, 431–446 (2004).

[PS1] Penkov, I., Serganova, V.: Cohomology of G/P for classical complex Lie supergroups G and characters of some atypical G-modules. Ann. Inst. Fourier (Grenoble) **39**, 845–873 (1989).

[PS2] Penkov, I., Serganova, V.: Characters of irreducible G-modules and cohomology of G/P for the Lie supergroup $G = Q(n)$. J. Math. Sci. **84**, 1382–1412 (1997).

[PS3] Penkov, I., Serganova, V.: The support of an irreducible Lie algebra representation. J. Algebra **209**, 129–142 (1998).

[PS4] Penkov, I., Serganova, V.: Generic irreducible representations of finite-dimensional Lie superalgebras. Internat. J. Math. **5**, 389–419 (1994).

[PS5] Penkov, I., Serganova, V.: Generalized Harish-Chandra modules. Moscow Math. J. **2**, 753–767 (2002).

[PS6] Penkov, I., Serganova, V.: Categories of integrable $\mathfrak{sl}(\infty)$-, $\mathfrak{o}(\infty)$-, $\mathfrak{sp}(\infty)$-modules. Contemp. Math. **557**, 335–357 (2011).

[PS7] Penkov, I., Serganova, V.: On bounded generalized Harish-Chandra modules. Ann. Inst. Fourier (Grenoble) **62**, 477–496 (2012).

[PS8] Penkov, I., Serganova, V.: Tensor representations of Mackey Lie algebras and their dense subalgebras. In: Mason, G., Penkov, I., Wolf, J. (eds.) Developments and Retrospectives in Lie Theory: Algebraic Methods, Developments in Mathematics, Vol. 38, pp. 291–330. Springer Verlag (2014).

[PS9] Penkov, I., Serganova, V.: Large annihilator category $\mathcal{O}$ for $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. J. Algebra **532**, 249–279 (2019).

[PSZ1] Penkov, I., Serganova, V., Zuckerman, G.: On the existence of $(\mathfrak{g}, \mathfrak{k})$-module s of finite type. Duke Math. J. **125**, 329–349 (2004).

[PSZ2] Penkov, I., Serganova, V., Zuckerman, G.: On categories of admissible $(\mathfrak{g}, \mathfrak{sl}(2))$-modules. Transform. Groups **23**(2), 463–489 (2018).

[PStr] Penkov, I., Strade, H.: Locally finite Lie algebras with root decomposition. Arch. Math. **80**, 478–485 (2003).
[PSty] Penkov, I., Styrkas, K.: Tensor representations of classical locally finite Lie algebras. In: Neeb, K. H., Pianzola, A. (eds.) Developments and trends in infinite-dimensional Lie theory, Progress in Mathematics, Vol. 288, pp. 127–150. Birkhäuser, Basel (2011).
[PZ1] Penkov, I., Zuckerman, G.: Generalized Harish-Chandra modules: a new direction of the structure theory of representations. Acta Appl. Math. **81**, 311–326 (2004).
[PZ2] Penkov, I., Zuckerman, G.: Generalized Harish-Chandra modules with generic minimal $\mathfrak{k}$-type, Asian J. Math. **8**, 795–812 (2004).
[PZ3] Penkov, I., Zuckerman, G.: A construction of generalized Harish-Chandra modules with arbitrary minimal $\mathfrak{k}$-type. Canad. Math. Bull. **50**, 603–609 (2007).
[PZ4] Penkov, I., Zuckerman, G.: A construction of generalized Harish-Chandra modules for locally reductive Lie Algebras. Transform. Groups **13**, 799–817 (2008).
[PZ5] Penkov, I., Zuckerman, G.: On the structure of the fundamental series of generalized Harish-Chandra modules. Asian J. Math. **16**, 489–514 (2012).
[PZ6] Penkov, I., Zuckerman, G.: On the existence of infinite-dimensional generalized Harish-Chandra modules. Sao-Paulo J. Math. **12**, 290–294 (2018).
[Po] Popescu, N.: Abelian categories with applications to rings and modules. Academic Press, London-New York,. London Mathematical Society Monographs, No. 3. (1973).
[Sch1] Scheunert, M.: The Theory of Lie Superalgebras. Lecture Notes in Math. **716**, Springer-Verlag, Berlin (1979).
[Sch2] Scheunert, M.: Invariant supersymmetric multilinear forms and the Casimir elements of P-type Lie superalgebras. J. Math. Physics **28**, 1180–1191 (1987).
[S1] Serganova, V.: Quasireductive supergroups. In: Boyallian, C., Galina, E., Saal, L. (eds.) New developments in Lie theory and its applications, Contemp. Math., Vol. 544, pp. 141–159. Amer. Math. Soc., Providence, RI (2011).
[S2] Serganova, V.: Classical Lie superalgebras at infinity. In: Gorelik, M., Papi, P. (eds.) Advances in Lie Superalgebras, Springer INdAM Ser. no. 7, pp. 181–201. Springer, Cham (2014).
[Vi] Vishnyakova, E.: On complex Lie supergroups and split homogeneous supermanifolds. Transform. Groups **16**, 265–285 (2011).
[Vo1] Vogan, Jr., D. A.: The Algebraic Structure of the Representations of Semisimple Lie Groups I. Ann. of Math. **109**, 1–60 (1979).
[Vo2] Vogan, Jr., D. A.: Representations of real reductive Lie groups. Progr. Math. **15**, Birkhauser, Boston (1981).
[Web] Webster, B.: Gelfand-Tsetlin modules in the Coulomb context. https://arxiv.org/abs/1904.05415.
[Wei] Weibel, C.: An introduction to homological algebra. Cambridge Univ. Press. (1994).
[Wey] Weyl, H.: The classical groups: their invariants and representations. Princeton University Press (1997).
[WZ] Willenbring, J., Zuckerman, G.: Small semisimple subalgebras of semisimple Lie algebras. In: Li, J.-S., Tan, E.-C., Wallach, N., Zhu, C.-B. (eds.) Harmonic analysis, group representations, automorphic forms and invariant theory. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., Vol. 12, pp. 403–429, World Scientific (2007).
[Zh] Zhelobenko, D.: Representations of reductive Lie algebras (in Russian). VO Nauka, Moscow (1994).
[Z] Zuckerman, G.: Generalized Harish-Chandra modules. In: Joseph, A., Melnikov, A., Penkov, I. (eds.) Highlights in Lie algebraic methods, Progr. Math., Vol. 295, pp. 123–143. Birkhauser (2012).

Index of Notation

Symbols
$(\cdot\,,\,\cdot)$ (bilinear form), 8, 34
$\langle\,,\,\rangle$, 162
$[\cdot\,,\,\cdot]$ (supercommutator), 11
$[\mathfrak{g},\mathfrak{g}]$ (derived subalgebra of $\mathfrak{g}$), 5
$[t,\,\cdot\,]$, 70

A
$A^!$ (left dual of A), 117
$A(\mathfrak{p}, E)$, 179
$\mathcal{A}_\mathfrak{g}$, $\mathcal{A}^r_\mathfrak{g}$, 111
$\mathcal{A}^r_\mathfrak{g}$-fmod, 111
$\mathcal{A}_\mathfrak{g}$-fmod, 111
A_∞ ($\mathfrak{sl}(\infty)$ root system), 34
a_λ, b_λ, c_λ, 90
$A_{m-1|n-1}$ ($\mathfrak{sl}(m|n)$ root system), 14
A_{n-1} ($\mathfrak{sl}(n)$ root system), 6
$\mathcal{A} = \{\mathcal{A}_n\}$, 213
$A_{n-1|n-1}$ ($\mathfrak{psl}(n|n)$ root system), 14
$\operatorname{ad} x$ (adjoint action), 5
$\operatorname{Aut} V$, 101
$\operatorname{Aut}_e \mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$, (automorphism group of $\mathfrak{g}_{\bar{0}}$), 22

B
B (Borel subgroup), 187
$\mathfrak{b}$ (Borel subalgebra), 7
$\mathfrak{b}$ (Borel subsuperalgebra), 26
B_∞, 34
$B_{\infty|2k}$, 45
$B_{m|n}$ ($\mathfrak{osp}(2m+1|2n)$ root system), 15
B_n ($\mathfrak{o}(2n+1)$ root system), 6
B_{red}, 200

C
$\mathbb{C}$ (complex numbers), 3
C (Grothendieck category), 105
$C(\mathfrak{t})$ (centralizer of the Lie subalgebra $\mathfrak{t}$), 72
$C = \{C_\kappa\}$ (chain of subspaces), 65
$\mathbb{C}[S_p]$, 90
$C_{1|n}$ ($\mathfrak{osp}(2|2n)$ root system), 15
C_∞ ($\mathfrak{sp}(\infty)$ root system), 34
C_n ($\mathfrak{sp}(2n)$ root system), 6
$\operatorname{ch}_\mathfrak{t} M$, 162

D
$D(2, 1; \alpha)$, 18
$\mathcal{D}(\infty)$ (algebra of polynomial differential operators), 148
D_∞, 34
$D_{\infty|2k}$, 45
$D_{m|n}$ ($\mathfrak{osp}(2m|2n)$ root system), 15
D_n ($\mathfrak{o}(2n)$ root system), 6
Δ (root system), 5, 22, 33
Δ^+, Δ^-, 25
$\Delta_\mathfrak{b}$, 59
$(\Delta_\mathfrak{h})^+_M$, $(\Delta_\mathfrak{h})^-_M$, 130
$(\Delta_\mathfrak{h})^F_M$, $(\Delta_\mathfrak{h})^I_M$, 130
$\operatorname{der}\mathfrak{g}$, 20

E
E_6, E_7, E_8, 5
$\operatorname{End} V$ (endomorphisms of V), 4
ext^i_A (ext-group), 111

F
$\mathcal{F}$ (generalized flag), 65
$\mathcal{F}(\mu)$, 148
F_4, 5
$F^\cdot(\mathfrak{p}, E)$, 163
$F^s(\mathfrak{p}, E)$, 164
$F_{3|1}$, 18
$f.$, 162
$F\ell(1, 2, \ldots, n-1, V)$, 188

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7

G
G (connected reductive algebraic group), 187
G_2, 5
$G_{2|1}$, 18
$\tilde{G}$, 101
$G := \varinjlim G_n$ (ind-group), 101
$\mathfrak{g}$ (Lie algebra), 3
$\mathfrak{g} = \varinjlim \mathfrak{g}_n$, 31
$\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ (Lie superalgebra), 11
$\mathfrak{g}_{\overline{0}}$ (even part of $\mathfrak{g}$), 11
$\mathfrak{g}[M]$ (Fernando-Kac subalgebra), 125
$\mathfrak{g}^\alpha$ (root space), 5, 22, 38
$\mathfrak{g}^\alpha$ (generalized root space), 37
$\mathfrak{g}^0_\mathfrak{k}$, 37
$\mathfrak{g}^+_M$, $\mathfrak{g}^-_M$, 131
$\mathfrak{g}^F_M$, $\mathfrak{g}^I_M$, 131
$\mathfrak{g}_n$, 31
G_{red}, 200
$\mathfrak{g}_{ss}$ (semisimple part of $\mathfrak{g}$), 3
$\mathfrak{gl}(\infty)$, 32
$GL(m|n)$, 200
$\mathfrak{gl}(m|n)$ (general linear Lie superalgebra), 12
$GL(n)$, 187
$\mathfrak{gl}(n)$ (general linear Lie algebra), 4
$GL(V)$, 188
$\mathfrak{gl}(V)$ (general linear Lie algebra), 4
$\mathfrak{gl}^M(V, V_*)$ (Mackey Lie algebra), 124

H
$\mathfrak{h}$ (Cartan subalgebra), 5
$\mathbf{H}(n)$, 19
$\mathfrak{h}$ (Cartan subsuperalgebra), 21
$\mathfrak{h}_n$ (Cartan subalgebra of $\mathfrak{g}_n$), 33

I
$\mathrm{Int}_\mathfrak{g}$, 99
$\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$, 156

K
$\kappa(x, y)$ (Killing form), 17

L
$\mathcal{L}$ (line bundle), 189
$L(i)$ ($\mathfrak{sl}(2)$-module), 132
$L_\mathfrak{b}(\lambda)$, 150
$L_B(\mu)$, 193
$\Lambda(\xi_1, \ldots, \xi_n)$ (Grassmann algebra), 19
λ (partition), 90
$\boldsymbol{\lambda}$ (standard Young tableau), 90
$|\lambda|$, 90
$\Lambda^2(V)$, 98
$\Lambda^p(V)$, 90
$\Lambda^{\frac{\infty}{2}}_{\mathbb{A}} V$, 145

M
$M(\mathfrak{p}, E)$, 179
M^+, 136
M^λ (weight space), 57
$M_\mathfrak{b}(\lambda)$, 150
M_{red} (reduced manifold), 199
$M_{r,s}$ ($r \times s$-matrices), 12
μ^+, μ^-, 107

N
$\mathfrak{n}$ (nilradical), 7
$N(a)$, 139
$N^\lambda_{\gamma,\lambda'}$ (Littlewood–Richardson coefficient), 92

O
$\mathcal{O}(-1)$, 189
$\mathfrak{o}(n)$ (orthogonal Lie algebra), 4
$\mathfrak{o}(V)$ (orthogonal Lie algebra), 4
$\overline{\mathcal{O}}_\mathfrak{b}$, 157
$\mathcal{O}_m$ (sheaf of superalgebras on M), 199
$\mathcal{O}_X$, 189
$\mathcal{OLA}$, 157
$\mathfrak{osp}(\infty|2k)$, 45
$\mathfrak{osp}(\infty|\infty)$, 45
$\mathfrak{osp}(m|\infty)$, 45
$\mathfrak{osp}(2m|2n)$ (orthosymplectic Lie superalgebra), 15
$\mathfrak{osp}(2m+1|2n)$ (orthosymplectic Lie superalgebra), 15
$\mathfrak{osp}(V)$ (orthosymplectic Lie superalgebra), 15
otr (odd trace), 16

P
P (parabolic subgroup), 187
$\mathbb{P}$ (projective space), 187
$\mathfrak{p}$ (parabolic subalgebra), 9
$\mathbf{P}(n)$, 200
$\mathbf{p}(V)$, 16
$\mathfrak{p}$ (parabolic subsuperalgebra), 29
$\mathfrak{p}_M$, $\mathfrak{p}^+_M$, $\mathfrak{p}^-_M$, 136
$\mathfrak{pgl}(V)$, 20
$\Phi_{i,j}$ (contraction map), 92
Pic X (Picard group), 189
Ψ_E (bottom layer map), 179
$\Psi_{i,j}$ (contraction map), 119
$\mathfrak{psl}(n|n)$, 12
$\mathfrak{psl}(V)$, 14
$\mathfrak{psq}(V)$, 17

Q
$Q(n)$, 200
$\mathfrak{q}(V)$, 16

R
$\mathfrak{r}_\mathfrak{g}$ (radical of $\mathfrak{g}$), 3
$R_\mathfrak{k}$ (functor), 122

rad X (radical of X), 113
$\rho_{f.}$, 162

S
$\mathbf{S}(n)$, 19
$\tilde{\mathbf{S}}(n)$, 19
$S^2(V)$, 98
$S^p(V)$, 90
$\mathcal{S}^B_{\mathbb{A}}$, 147
$\mathcal{S}^D_{\mathbb{A}}$, 147
$S^\infty_{\mathbb{A}} V$, 145
$S^\infty_{\mathbb{A}} V_*$, 145
$S_{\mathfrak{k}}$ (functor), 122
$\mathbb{S}_\lambda(V)$, 90
S_p, 90
$\mathfrak{sl}(\infty, \infty)$, 45
$\mathfrak{sl}(\infty|n)$, 45
$\mathfrak{sl}(\infty)$, 32
$\mathfrak{sl}(m|n)$ (special linear Lie superalgebra), 12
$SL(n)$, 187
$\mathfrak{sl}(n)$ (special linear Lie algebra), 4
$\mathfrak{sl}(V)$ (special linear Lie algebra), 4
$\mathfrak{o}(\infty)$, 35
$SO(n)$, 187
soc M (socle of M), 91
$\operatorname{soc}^i M$, 91
$\overline{\operatorname{soc}}^i M$ (layer of socle filtration), 91
$SOSp(m|2n)$, 200
$Sp(2n)$, 187
$\mathfrak{spb}(\infty)$, 45
$\mathfrak{sp}(\infty)$, 35
$\mathfrak{sp}(n)$ (symplectic Lie algebra), 4
$\mathfrak{spb}(V)$, 16
$\mathfrak{sp}(V)$ (symplectic Lie algebra), 4
$\mathfrak{sq}(\infty)$, 45
$\mathfrak{sq}(V)$, 16
$\mathrm{St}_{\mathcal{F}}$, 83
str (supertrace), 12
$\mathrm{Stz}(\mathfrak{s})$, 49
$\sum_{\mathfrak{b}}$ (set of simple roots), 8, 26
$\operatorname{supp}_{\mathfrak{t}} V$, 72

T
$T(V)$, 100
$T^{p,q}$, 89
$T^{\{p,q\}}$, 93
$T^{\leq r}$, 100
$\mathbb{T}_{\mathfrak{g}}$ (category of tensor modules), 99

U
$U(\mathfrak{k})$ (universal enveloping algebra), 72
$U^\perp$, 81

V
V (vector space), 4
V^* (dual space), 4
$V^{\otimes p} \otimes V_*^{\otimes q}$, 89
V_* (restricted dual), 56
$\tilde{V}_{\lambda,\mu}$, 92
$V_{\lambda,\mu}$, 92
V_n, 34

W
W (Weyl group), 9
$\mathbf{W}(n)$, 19

X
$X_{\mathfrak{sl}}(\mu)$, 148

Y
Y_λ, 90

Z
$\mathbb{Z}_2$, 12
$Z_{\mathfrak{g}}$ (center of $\mathfrak{g}$), 4, 5

Index

Symbols
$(\mathfrak{g}, \mathfrak{k})$-module, 127, 159
$(\mathfrak{g}, \mathfrak{k})$-regular, 163

A
Absolute weight module, 100
Adjacent Borel subsuperalgebras, 28
Adjacent Borel subsupergroups, 200
Admissible $(\mathfrak{g}, \mathfrak{k})$-module, 159
Antidominant weight, 193, 203, 214

B
Borel subalgebra, 7, 56
Borel subgroup, 187
Borel subsuperalgebra, 26
Borel subsupergroup, 200
Bott–Borel–Weil Theorem, 193
Bottom layer map, 179
Bottom layer subquotient, 179

C
Cartan subalgebra, 5, 78
Cartan subsuperalgebra, 21
Cartan-type Lie superalgebra, 19
Category of tensor modules, 104
Chain of subspaces, 65
Chevalley's Theorem, 100
Classical Lie superalgebra, 17
Classical locally simple Lie algebra, 35
Classical locally simple Lie superalgebra, 45
Classically semisimple Lie superalgebra, 43
Closed generalized flag, 81
Closed subspace, 81
Coherent family, 140
Compatible basis, 63, 65
Compatible parabolic subalgebra, 163
Connected by an odd reflection, 28, 200
Connected reductive algebraic group, 187
Conormal bundle, 199
Coordinate space, 63
Cuspidal module, 138

D
Defining representation, 205
Degenerate odd reflection, 28
Demazure's Lemma, 206
Diagonal inclusion, 47
Diagonal locally simple Lie algebra, 48
Dominant simple P-module, 220
Dominant weight, 193, 202, 203, 214
Dual system, 77
Dual system of one-dimensional spaces, 77
Dynkin Borel subalgebra, 61

E
Equivalent supernatural numbers, 48
Essential extension of a module, 105
Even root, 23
Exhaustion, 31
Exhaustive filtration, 91

F
Fernando–Kac subalgebra, 126
Finite corank Lie subalgebra, 102
Finite-integrable module, 144
Finite Loewy length, 91
Finite type $(\mathfrak{g}, \mathfrak{k})$-module, 159
First Reconstruction Theorem, 169
Flag, 63, 64
Fundamental series of $(\mathfrak{g}, \mathfrak{k})$-modules, 163

G
$\tilde{G}$-invariant module, 102
General linear Lie superalgebra, 12

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2022

I. Penkov, C. Hoyt, *Classical Lie Algebras at Infinity*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-89660-7

Generalized $\mathfrak{h}$-root decomposition, 22, 38, 42
Generalized $\mathfrak{h}$-weight space, 38
Generalized flag, 65
Generalized Harish-Chandra module, 159
Generalized root decomposition, 22
Generalized root space, 38
Generalized weight module, 128
General linear Lie algebra, 4
Generic $\mathfrak{k}$-type, 163
$GL(V)$-linearized line bundle, 191

H
Harish-Chandra module, 160
Height of a Borel subsupergroup, 210

I
Index of a Lie subalgebra, 48
Injective module, 105
Isotropic generalized flag, 67

K
$\mathfrak{k}$-locally finite vector, 127
$\mathfrak{k}$-type, 163
Kac–Cheng Theorem, 21
Kac module, 202
Koszul dual of a quadratic algebra, 117
Koszul ring, 110

L
Layers of socle filtration, 91
Left dual of a quadratic algebra, 117
Levi component, 3
Levi–Malcev Theorem, 3
Lie superalgebra, 11
Lie supergroup, 200
Line bundle, 189, 212
Line subalgebra, 134
Locally $\mathfrak{k}$-finite module, 72
Locally finite action, 99, 125
Locally finite Lie algebra, 31
Locally nilpotent action, 37
Locally nilpotent Lie algebra, 37
Locally reductive Lie algebra, 77
Locally simple G-module, 212
Locally solvable Lie algebra, 37
Locally solvable radical, 37
Loewy length, 91

M
Mackey closed subspace, 81
Mackey's Theorem, 124
Marks of a weight, 201
Maximal generalized flag, 65
Minimal $\mathfrak{k}$-type, 163
Minimal compatible parabolic subalgebra, 163
Module over a Lie supergroup, 205
Module over an ind-group, 212
Multiplicity free, 139
Multiset, 162

N
Natural $\mathfrak{sl}(\infty)$-module, 56
Natural representation, 205
Nilradical, 179
Non-splitting supermanifold, 200

O
Odd reflection, 28
Odd root, 23
Odd trace, 16
Ordered Grothendieck category of $\mathfrak{g}$-modules, 105
Orthogonal Lie algebra, 4
Orthosymplectic Lie superalgebra, 15

P
Parabolic subalgebra, 9
Parabolic subgroup, 187
Parabolic subsuperalgebra, 29
Parabolic subsupergroup, 200
Parity changing functor, 199
Partition, 90
Π (parity changing functor), 199
Picard group, 189
Primal Lie subalgebra, 173
Projectivization, 187
Pro-rational G-module, 212
Pseudo highest weight module, 150

R
Radical filtration, 113
Radical of a Lie algebra, 3
Radical of a module, 113
Rank of a vector bundle, 212
Rational G-module, 212
Reduced manifold, 199
Reducible pair, 161
Reductive Lie algebra, 5
Regular weight, 202
Representation of a Lie supergroup, 205
Root, 5, 22, 33, 38, 42, 130
Root injection, 32
Root isomorphism, 34
Root-reductive Lie algebra, 32
Root space, 5, 22, 38
Root subalgebra, 7
Root system, 5, 22, 33

S
Scheunert's Theorem, 21
Schur functor, 90
Second Reconstruction Theorem, 169

Semiclosed generalized flag, 83
Semi-infinite fundamental representation, 145
Semi-infinite set, 145
Semisimple Lie algebra, 3, 37
Semisimple Lie superalgebra, 13, 42
Semisimple part, 3, 13
Serganova functor, 177
Shadow decomposition, 131
Simple Lie algebra, 3
Simple Lie superalgebra, 13
Simple root, 8, 26, 60
Skornyakov's Odd Reflection Lemma, 206
$\mathfrak{sl}(2)$-type root, 134
Socle filtration, 91
Socle of a module, 91
Solvable radical, 3
Special linear Lie algebra, 4
Special linear Lie superalgebra, 12
Splittable Lie subalgebra, 78
Splitting Borel subalgebra, 57
Splitting Borel subgroup, 211
Splitting Cartan subalgebra, 78
Splitting parabolic subalgebra, 67, 211
Splitting parabolic subgroup, 211
Splitting supermanifold, 200
Standard Borel subsuperalgebra, 29
Standard set of simple roots, 29
Standard Young tableau, 90
Steinitz number, 48
Strange Lie superalgebra, 17
Strict $(\mathfrak{g}, \mathfrak{k})$-module, 173
Submultiset, 162
Superalgebra, 199
Supercommutator, 11
Superderivation, 19, 20
Supermanifold, 199
Supernatural numbers, 48
Supertrace, 12
Supertranspose, 12
Symplectic Lie algebra, 4

T
$\mathfrak{t}_0$-compatible parabolic subalgebra, 178
Taut pair of semiclosed generalized flags, 83
Tensor module, 100
Toral subalgebra, 5, 70
Triangular decomposition, 26
Typical weight, 202

U
Underlying manifold, 199

V
Vector bundle, 199, 212
Vogan norm, 163

W
Weight, 201
Weight module, 57, 128
Weight space, 128
Weyl group, 9

Z
Zuckerman functor, 162